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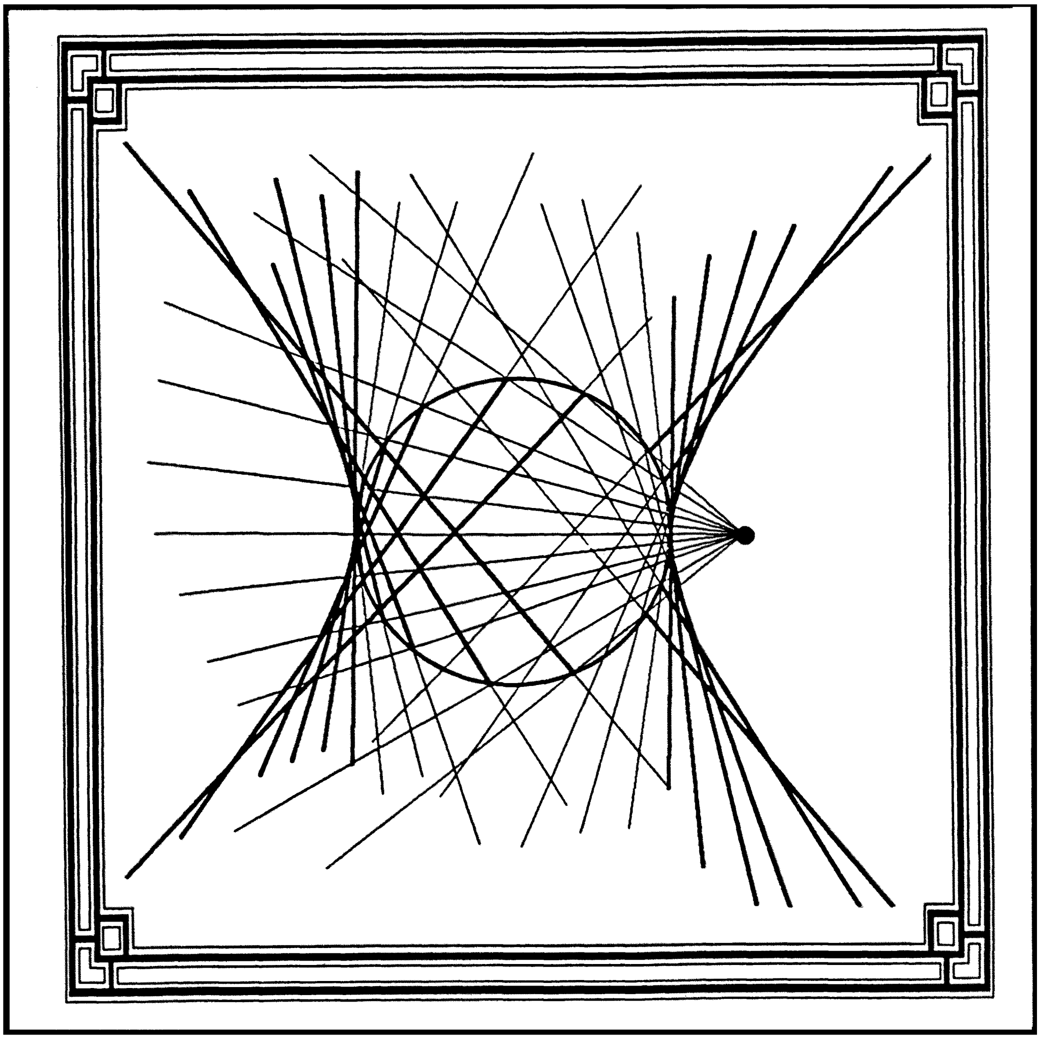
MATHEMATICS MAGAZINE



- Central Points and Central Lines in the Plane of a Triangle
- The Conics of Ludwig Kiepert: A Comprehensive Lesson
- Museum Exhibits for the Conics

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AUTHORS

Clark Kimberling is profiled in the promotional piece at the University of Evansville as "its own Renaissance man." Between ventures in number theory and geometry, he enjoys composing for university choirs, performing on recorders, and writing history, including a couple of publications about Emmy Noether. He first got hooked on triangle geometry during studies of probabilistic metric spaces at Illinois Institute of Technology, where he received his Ph.D. in functional equations under Abe Sklar in 1979. He treats triangle centers (centroid, incenter, circumcenter, etc.) as functions over a suitable space of triangles, rather than mere two-dimensional points, and, like the ancient Pythagoreans, he thinks there is something magic about these centers.

Roland Eddy began his teaching career in the Newfoundland high school system. In 1968, he was appointed to Memorial University as a lecturer responsible for freshman mathematics classes. In 1972, he completed a Master's thesis on random number generators but was encouraged by his then department head W. J. Blundon to turn his hand to geometry, in particular, geometric inequalities. He became an active researcher in this and other areas in geometry, attaining the rank of Professor of Mathematics in 1990. He and Rudolf Fritsch began collaborative work in the geometry of the triangle, a subject dear to both of their hearts, in 1985 and continue to this day.

Rudolf Fritsch received his Ph.D. at the Universität des Saarlandes with a thesis on simplicial sets in 1968 and the venia legendi for mathematics at the Universität Konstanz in 1973. He began his career as a high school teacher in the late sixties, which culminated in his being appointed full professor for mathematics, in particular mathematical education, at the Ludwig—Maximilian–Universität in München in 1981. His mathematical interests include algebraic topology, categories and functors, foundations of geometry, history of mathematics, and mathematical education, emphasizing a stronger consideration of spatial aspects. While collaborating at Memorial University with R. Piccinini on the monograph *Cellular Structures in Topology*, he met Roland Eddy. They discovered a common love for elementary geometry, which led to the paper on the Kiepert Conics.

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ARTICLES

Central Points and Central Lines in the Plane of a Triangle

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1. Introduction

Triangle geometry ranks among the most enduring topics in all of mathematics. A treasury of triangle lore abounds in Euclid's *Elements* of 2.3 millenia ago, and still today interesting elementary configurations are being discovered. To sample current interests in triangle geometry, one may page through the voluminous *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić, and Volenec [**36**], the Canadian Mathematical Society's problem-solving journal *Crux Mathematicorum*, or the journals of the Mathematical Association of America.

Central to triangle geometry are certain special points, or *centers*. Unlike the circle and rectangle, a triangle has more than one center, as indicated by the names of the four ancient centers: *centroid*, *incenter*, *circumcenter*, and *orthocenter*, as well as the *isogonic* (equal-angle) *center*, which Eves [**15**, p. 84], describes as the first center discovered after ancient times.

Several remarkable centers have debuted after 1980, and new ones will undoubtedly appear during the next few years. In this article, I shall list (Tables 1 and 2) and discuss the centers I have found in the literature and a few others. The article is in six sections: (1) Introduction, (2) Trilinear Coordinates, (3) Regular Centers, (4) More Centers, (5) Constant Distance-Ratios, and (6) Topics for Undergraduate Research.

Among the references listed at the end of this article, Altshiller-Court [**1**] and Johnson [**22**] provide excellent introductions to triangle geometry but do not mention trilinear coordinates, which play an essential role in this treatment and add an algebraic twist to the subject.

2. Trilinear Coordinates

The triangle is labeled in the usual (Euler's) manner: vertices A, B, C; angles A, B, C; and sidelengths a, b, c: $a = |BC|, b = |CA|, c = |AB|$. If X is a point in the plane of ABC, then its position is completely determined by the *ratios* of directed distances from X to the sidelines. Such ratios can therefore serve as coordinates for X. Accordingly, any ordered triple (α, β, γ) of numbers respectively proportional to the directed distances from X to the sidelines BC, CA, AB are called *homogeneous trilinear coordinates*, or, less formally, *trilinears*. Since ratios, rather than the actual distances, are used, the triple (α, β, γ) is written with colons, like this: $\alpha : \beta : \gamma$. For example, the vertices of the reference triangle ABC are represented very simply as $A = 1:0:0, B = 0:1:0, C = 0:0:1$. The incenter, being equidistant from the sides, is $1:1:1$. The centroid is $1/a : 1/b : 1/c$, alias $\csc A : \csc B : \csc C$, alias $bc : ca : ab$.

The *actual* directed distances from a point $X = \alpha : \beta : \gamma$ to the respective sidelines, called *actual trilinear distances*, are $k\alpha, k\beta, k\gamma$, where $k = 2\Delta/(a\alpha + b\beta + c\gamma)$ and Δ = area of ABC. The point X lies inside ABC if, and only if, all three numbers are positive or all negative. For example, the incenter, $1:1:1$ $(= -1:-1:-1)$ lies inside ABC, unlike the three excenters, $-1:1:1$, $1:-1:1$, and $1:1:-1$, as shown in FIGURE 9.

Definition of center The captions for Figures 1–4 indicate that triangle centers are functions of the sidelengths, or equivalently, the angles. In order to crystalize this notion, let $\mathbb{T}$ be the set of all triples (a, b, c) of real numbers that are sidelengths of a triangle. That is,

$$\mathbb{T} = \{(a,b,c): 0 < a < b + c, 0 < b < c + a, 0 < c < a + b\}.$$

On any subset $\mathbb{U}$ of $\mathbb{T}$, define a *center-function* as a nonzero function $f(a,b,c)$ homogeneous in a, b, c (i.e., $f(ta, tb, tc) = t^n f(a,b,c)$ for some nonnegative integer n, all $t > 0$, and all (a,b,c) in $\mathbb{U}$) and symmetric in b and c (i.e., $f(a,c,b) = f(a,b,c)$ for all (a, b, c) in $\mathbb{U}$). A *center on* $\mathbb{U}$ is an equivalence class $\alpha : \beta : \gamma$ of ordered triples (α, β, γ) given by

$$\alpha = f(a,b,c), \beta = f(b,c,a), \gamma = f(c,a,b)$$

for some center-function f defined on $\mathbb{U}$, where two triples $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are equivalent if $\alpha_1 \beta_2 = \alpha_2 \beta_1$ for all (a, b, c) in $\mathbb{U}$. (Usually $\mathbb{U} = \mathbb{T}$, but with the extra trouble, we have now also accommodated centers such as the *Feuerbach point*, which fails to exist when $a = b = c$.)

The fact that this definition of "center" is algebraic may come as a surprise. Let's look into the connections that the three algebraic requirements (homogeneity, symmetry in b and c, and cyclic substitution) have with geometric properties shared by familiar examples of triangle centers. First, take a glance ahead to Table 1, specifically, at the algebraic form of the coordinate $\boldsymbol{\alpha}$ for the first few centers. The three algebraic requirements are easily seen to hold. For example, the circumcenter, X_3, has $\alpha(a,b,c) = a(b^2 + c^2 - a^2)$, which is homogeneous of degree 3, satisfies $a(a,c,b) = \alpha(a,b,c)$, and has second coordinate $\alpha(b,c,a) = b(c^2 + a^2 - b^2)$ and third coordinate $\alpha(c,a,b)$.

Homogeneity ensures that similar triangles have similarly situated centers. For, if a triangle $T = A'B'C'$ is similar to the reference triangle ABC, then for some $t > 0$ the sidelengths of T are ta, tb, tc. Suppose X is a center, and let f be a center function for X. Then the value of X at ABC is $f(a,b,c):f(b,c,a):f(c,a,b)$, and the value of X at T is $f(ta,tb,tc):f(tb,tc,ta):f(tc,ta,tb)$, which by homogeneity equals $f(a,b,c):f(b,c,a):f(c,a,b)$. That is, the *ratios* of distances from X to the sidelines remain unchanged.

Next, consider the geometric meaning of cyclic substitution. Perhaps you remember from an early age a certain fascination with the way you keep doing the "same thing in different places" in order to construct the centroid, incenter, or circumcenter. For the centroid, for example, you first draw a line from A to the midpoint of B and C, and then repeat, but this time from B to the midpoint of C and A—and there you have it: first A, B, C and then B, C, A, followed, if you wish, by C, A, B.

Finally, the symmetry of $f(a,b,c)$ in b and c corresponds to interchanging the roles of B and C (or b and c) when carrying out a construction relative to the vertices in the order A, B, C. For example, when drawing the median from A to the midpoint of BC, you get the same thing if you go from A to the midpoint of CB.

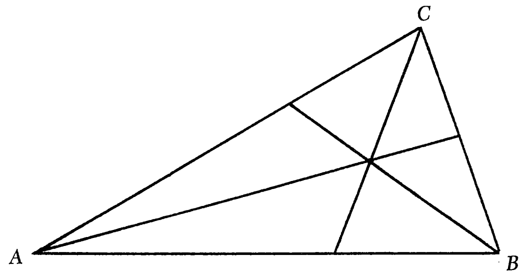

FIGURE 1

Angle bisectors meeting at incenter α: β: $\gamma = 1$: 1: 1.

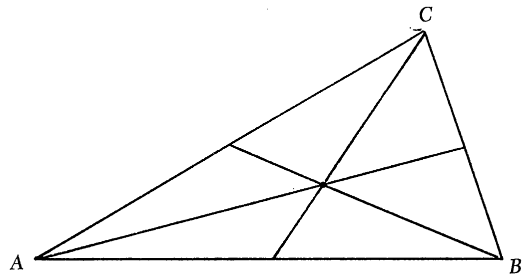

FIGURE 2

Medians meeting at centroid α: β: $\gamma = 1/a$: $1/b$: $1/c$.

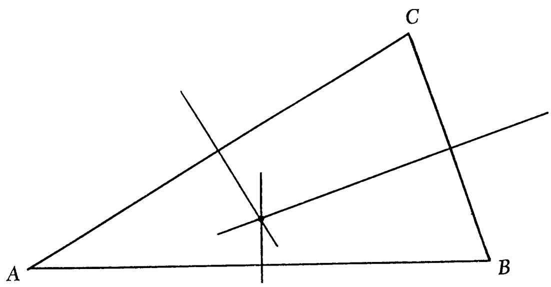

FIGURE 3

Perpendicular bisectors meeting at circumcenter α: β: $\gamma = \cos A$: $\cos B$: $\cos C$.

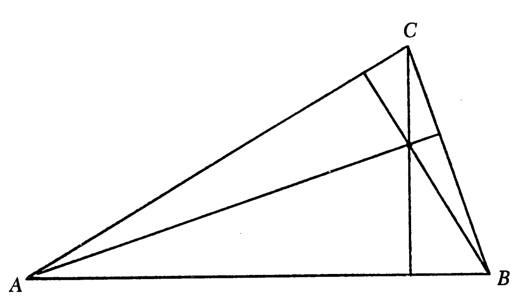

FIGURE 4

Altitudes meeting at orthocenter α: β: $\gamma = \sec$ A: $\sec B$: $\sec C$.

Isogonal conjugates The *isogonal conjugate* of a point X in the plane of ABC, but not on a sideline, BC, CA, or AB, is constructed as follows: Reflect the line AX about the angle bisector at A, and reflect BX and CX about the angle bisector at B and at C, respectively. The three reflected lines concur in the isogonal conjugate of X, as indicated in FIGURE 5.

The use of the symbol X^{-1} for isogonal conjugate corresponds to the fact (e.g. [**1**, p. 273]) that if $X = \alpha : \beta : \gamma$, then $X^{-1} = \alpha^{-1} : \beta^{-1} : \gamma^{-1}$.

Harmonic conjugates Suppose W, X, Y, Z are collinear points in the plane of ABC. Then Y and Z are *harmonic conjugates with respect to* W *and* X if $|WY|/|YX| = |WZ|/|ZX|$, that is, if Y and Z divide the interval WX internally and externally in the same ratio. To construct Z from given collinear points W, X, Y, let X' be a point not on WX and let W' be any point on segment YX' lying strictly between Y and X'. Let $Y' = WX' \cap XW'$ and $Z' = WW' \cap XX'$. Then $Z = WX \cap Y'Z'$, as shown in FIGURE 6.

Now, if $W = \alpha : \beta : \gamma$ and $X = \alpha' : \beta' : \gamma'$ are any two points, it is not hard to show using actual trilinear distances and similar triangles that the points

$$Y = \alpha + \alpha' : \beta + \beta' : \gamma + \gamma' \text{ and } Z = \alpha - \alpha' : \beta - \beta' : \gamma - \gamma' \tag{1}$$

are harmonic conjugates with respect to W and X.

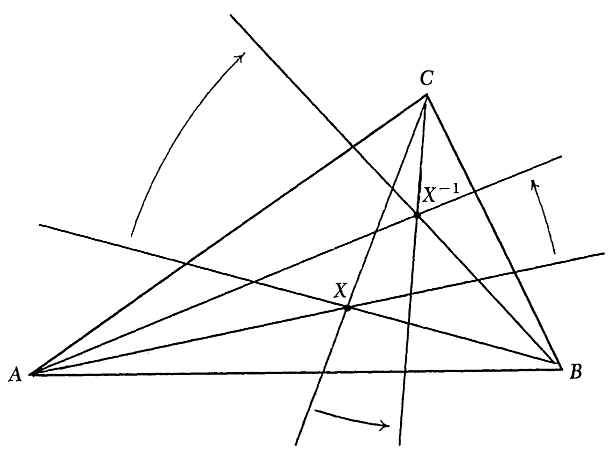

FIGURE 5
Lines AX, BX, CX are reflected about the angle bisectors and concur in the isogonal conjugate, X^{-1}, of X.

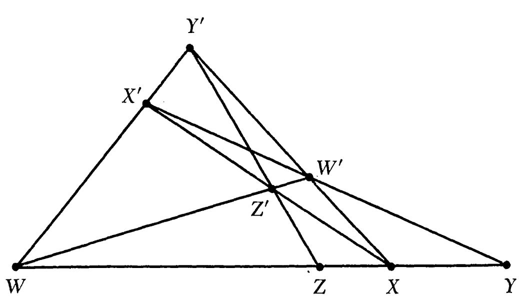

FIGURE 6
Y and Z are harmonic conjugates with respect to W and X. Given W, X, and Y (or Z), this figure shows how to construct Z (or Y).

Special triangles Certain triangles associated with the reference triangle ABC have well-established names. We list five here and refer to them in the sequel:

medial triangle: the triangle $A'B'C'$ whose vertices are the midpoints of BC, CA, AB;

$$A' = 0\colon 1/b\colon 1/c \qquad B' = 1/a\colon 0\colon 1/c \qquad C' = 1/a\colon 1/b\colon 0$$

anticomplementary triangle: the triangle $A'B'C'$ whose medial is the reference triangle;

$$A' = -1/a\colon 1/b\colon 1/c \qquad B' = 1/a\colon -1/b\colon 1/c \qquad C' = 1/a\colon 1/b\colon -1/c$$

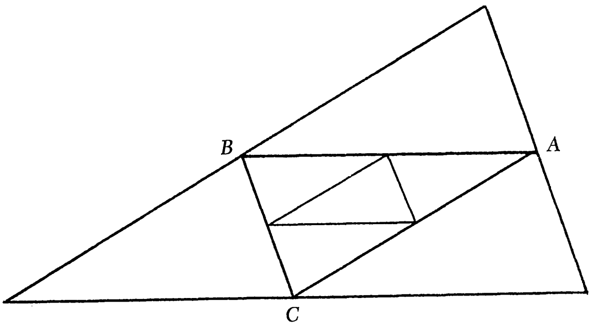

FIGURE 7
The triangle inscribed in ABC is the medial; the triangle circumscribing ABC, the anticomplementary.

orthic triangle: the triangle $A'B'C'$ whose vertices are the feet of the altitudes of ABC;

$$A' = 0\colon \sec B\colon \sec C \qquad B' = \sec A\colon 0\colon \sec C \qquad C' = \sec A\colon \sec B\colon 0$$

tangential triangle: the triangle $A'B'C'$ formed by lines tangent to the circumcircle at the vertices of ABC;

$$A' = -a : b : c \qquad B' = a : -b : c \qquad C' = a : b : -c$$

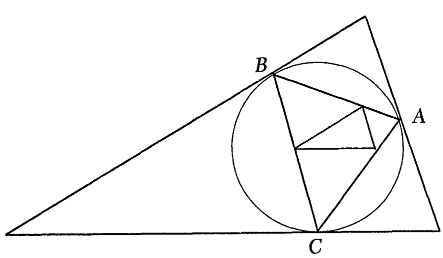

FIGURE 8
The triangle inscribed in ABC is the orthic; the triangle circumscribing ABC, the tangential.

excentral triangle: the triangle $A'B'C'$ whose vertices are the excenters of ABC ($A'B'C'$ is also called the *tritangent triangle*);

$$A' = -1:1:1 \qquad B' = 1:-1:1 \qquad C' = 1:1:-1.$$

Algebraically, the coordinates $-1:1:1$ define the A-excenter. Usually, of course, it is defined geometrically as the center of the A-excircle, which is one of the four circles tangent to all three sides of ABC—namely the one whose only point of tangency with ABC lies between B and C. See FIGURE 9.

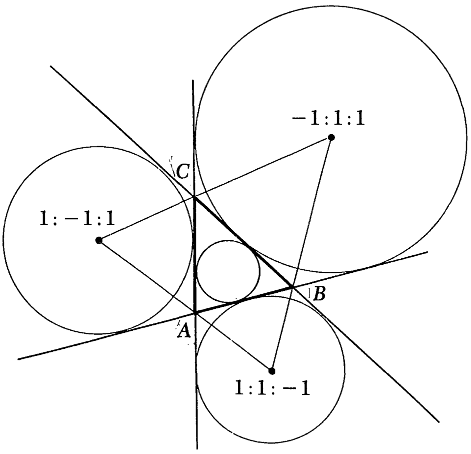

FIGURE 9
Triangle ABC with incircle, the three excircles, and the excentral triangle circumscribing ABC.

Perspective and homothetic triangles Triangles $A'B'C'$ and $A''B''C''$ for which the lines $A'A''$, $B'B''$, $C'C''$ concur are *perspective triangles*, and the point of concurrence is the *center of perspective* of the two triangles. According to Desargue's Theorem, the points $B'C' \cap B''C''$, $C'A' \cap C''A''$, $A'B' \cap A''B''$ are collinear; their line is the *axis of perspective*. See FIGURE 10.

If $B'C'$ and $B''C''$ are parallel, and the other two matching pairs of sides are also parallel, the triangles are not only perspective, but *homothetic*, and their center of perspective is called their *homothetic center*. In this case, the axis of perspective is the *line at infinity*, discussed below. (I chose the terminology "perspective triangles" over the more common "triangles in perspective", because, well, who ever says "triangles in similarity", or "triangles in homothety"?)

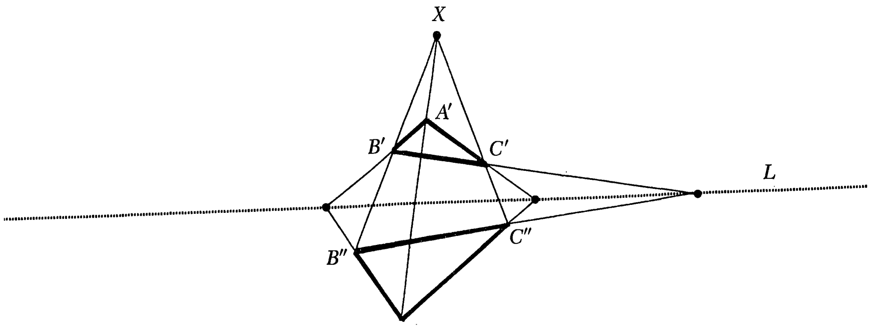

FIGURE 10
Perspective triangles $A'B'C'$ and $A''B''C''$ with center of perspective, X, and axis of perspective, L.

Pedal and antipedal triangles The *pedal triangle* of a point P is the triangle $T = A'B'C'$ whose vertices are the feet of the perpendiculars from P to the sidelines of ABC. If $P = \alpha : \beta : \gamma$, then

$$A' = 0 : \beta + \alpha \cos C : \gamma + \alpha \cos B$$
$$B' = \alpha + \beta \cos C : 0 : \gamma + \beta \cos A$$
$$C' = \alpha + \gamma \cos B : \beta + \gamma \cos A : 0.$$

The *antipedal triangle* of P is the triangle $\hat{T} = A'B'C'$ formed by the line through A perpendicular to PA, the line through B perpendicular to PB, and the line through C perpendicular to PC:

$$A' = -(\beta + \alpha\cos C)(\gamma + \alpha\cos B) : (\gamma + \alpha\cos B)(\alpha + \beta\cos C) : (\beta + \alpha\cos C)(\alpha + \gamma\cos B)$$
$$B' = (\gamma + B\cos A)(\beta + \alpha\cos C) : -(\gamma + \beta\cos A)(\alpha + \beta\cos C) : (\alpha + \beta\cos C)(\beta + \gamma\cos A)$$
$$C' = (\beta + \gamma\cos A)(\gamma + \alpha\cos B) : (\alpha + \gamma\cos B)(\gamma + \beta\cos A) : -(\alpha + \gamma\cos B)(\beta + \gamma\cos A).$$

The triangles $T(P)$ and $\hat{T}(P^{-1})$ are homothetic, and the product of their areas equals the square of the area of ABC; Gallatly [**19**, pp. 56–58] proves these charming theorems and then derives trilinears for the homothetic center.

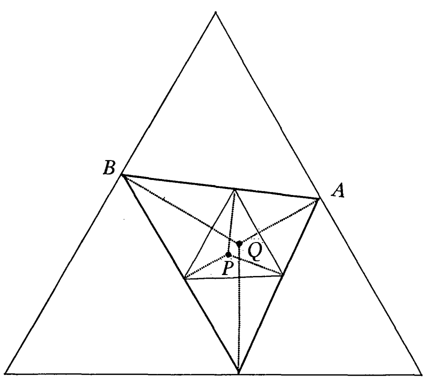

FIGURE 11

Pedal triangle of point P inscribed in ABC and antipedal triangle of Q circumscribing ABC. (See the notes at X_{13} and X_{15} in Table 1.)

Euler line Lines that pass through pairs of centers are frequently encountered in the literature. The most common of these is the *Euler line*, which contains the centroid, circumcenter, and 13 other centers described in Table 1. The Euler line will be discussed again in Section 5. For now, we note that it consists of all the points $\alpha : \beta : \gamma$ that satisfy

$$\alpha \sin 2A \sin(B - C) + \beta \sin 2B \sin(C - A) + \gamma \sin 2C \sin(A - B) = 0.$$

Line at infinity and the circumcircle A remarkable thing happens if you pick any point X on the circumcircle (the circle that passes through A, B, C) and try to locate the isogonal conjugate of X: The three reflected segments that are supposed to concur in X^{-1} turn out to be parallel! However, nothing is lost—indeed, much is gained—if you consider these parallel lines to meet infinitely far away from ABC. If X is allowed to vary around the circumcircle, then X^{-1} varies through what is called the *line at infinity*. An equation for this line is $\alpha a + \beta b + \gamma c = 0$ [**5**, p. 619]. Using this equation, you can quickly confirm that an equation for the circumcircle is $\beta\gamma a + \gamma\alpha b + \alpha\beta c = 0$. An example of a center on the line at infinity is

$$\cos A - \cos B \cos C : \cos B - \cos C \cos A : \cos C - \cos A \cos B.$$

This center (X_{30} in Table 1) is also on the Euler line. Its isogonal conjugate lies on the circumcircle.

Lines in general The line joining arbitrary points $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ is the set of points $\alpha : \beta : \gamma$ satisfying

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0. \tag{2}$$

When regarding points as centers (i.e., functions of the variable (a, b, c), not just points in a plane), we take the preceding sentence, in the case that $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ are centers, to define a *central line*. The following theorem is well known. Three lines

$$l_1\alpha + m_1\beta + n_1\gamma = 0, \qquad l_2\alpha + m_2\beta + n_2\gamma = 0, \qquad l_3\alpha + m_3\beta + n_3\gamma = 0$$

concur in a point if, and only if,

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0; \tag{3}$$

this equation can be used to confirm the many concurrences recorded in Section 3.

Two lines, as functions of the variable (a, b, c), are *parallel* if they are parallel in the usual sense for each particular triangle (a, b, c). Equivalently, and for purposes of computing and algebraic proofs, lines $l\alpha + m\beta + n\gamma = 0$ and $l'\alpha + m'\beta + n'\gamma = 0$ are, as in [**5**, p. 620], *parallel* if

$$a(mn' - nm') + b(nl' - ln') + c(lm' - ml') = 0 \tag{4}$$

for all (a, b, c) in $\mathbb{T}$, and *perpendicular* if

$$2abc(ll' + mm' + nn') - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0 \tag{5}$$

for all (a, b, c) in $\mathbb{T}$. For discussions of perpendicular lines in terms of trilinears, see [**33**] and [**46**].

Brocard angle, ω There appear to be only two well-known notable points of a triangle whose coordinates satisfy homogeneity and cyclic substitution but not symmetry. They are the *Brocard points*, b/c: c/a: a/b and c/b: a/c: b/a. The first is the interior point Z of ABC for which the angles ZAB, ZBC, ZCA are equal. The common angle is the *Brocard angle*, ω, given by

$$\cot\omega = \cot A + \cot B + \cot C.$$

The second Brocard point is the interior point Z of ABC for which the angles ZAC, ZCB, ZBA are equal. Again, the common angle is ω. Less well known are the first and second *Yff points*, constructed in [**45**] in a manner analogous to the construction of the Brocard points.

3. Regular Centers

Let $\sqrt{\mathscr{P}} = (1/4)\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$, the area of triangle ABC. A center X is a *regular center* if there exists a function $f(a,b,c)$ that is a polynomial in $a, b, c, \sqrt{\mathscr{P}}$ such that $X = f(a,b,c)$: $f(b,c,a)$: $f(c,a,b)$.

In Table 1, 101 centers are given names of the form X_i, followed by a verbal name if known, usually followed by an equation of the form $\boldsymbol{\alpha} = f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ or $\boldsymbol{\alpha} = g(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, from which trilinear coordinates can be written out by cyclically permuting a, b, c or A, B, C. For example, for X_3, you see $\boldsymbol{\alpha} = \cos \boldsymbol{A}$, signifying that $\alpha : \beta : \gamma = \cos A$: $\cos B$: $\cos C$. For some centers, two choices of $\boldsymbol{\alpha}$ appear; for X_3, the second $\boldsymbol{\alpha}$ is $\boldsymbol{a}(\boldsymbol{b}^2 + \boldsymbol{c}^2 - \boldsymbol{a}^2)$, obtained as follows:

$$\cos A : \cos B : \cos C = \frac{b^2+c^2-a^2}{2bc} : \frac{c^2+a^2-b^2}{2ca} : \frac{a^2+b^2-c^2}{2ab}$$
$$= a(b^2+c^2-a^2) : b(c^2+a^2-b^2) : c(a^2+b^2-c^2).$$

In some cases it is more convenient to express α in terms of A, B, C than a, b, c, but by the Law of Cosines and the Law of Sines, each $g(A, B, C)$ in Table 1 can be written as an $f(a, b, c)$.

Note that the isogonal conjugate of a regular center is also a regular center; for, if $X = f(a,b,c)$: $f(b,c,a)$: $f(c,a,b)$, then

$$X^{-1} = f(b,c,a)f(c,a,b) : f(c,a,b)f(a,b,c) : f(a,b,c)f(b,c,a).$$

Moreover, if W, X are regular centers, then any harmonic conjugates Y, Z given as in (1) are obviously regular centers.

References cited in Table 1 Derivations of trilinears for certain centers are hard to find in the literature. To address this problem, for each center, a reference number may appear in square brackets, just after the $\boldsymbol{\alpha}$-equation(s). For example, "**[5, p. 622]**" following "$\boldsymbol{\alpha} = \mathbf{1}$" at $\mathbf{X_1}$ signifies that reference **[5]** contains a derivation of the trilinears of the incenter. Another reference number may appear a bit later, documenting further properties of the center, but if not, you can probably find details in Altshiller-Court [1] or Johnson [22]. These books both cover the essentials of triangle geometry, yet each contains a wealth of information not found in the other. (The latter describes the former as "a recent and successful American text with which we expect to enter into friendly competition.")

Central lines Every center is the point of concurrence of many central lines. In order to list many such constructions in Table 1, I use the notation $L(i, j)$ for the line of centers X_i and X_j. For example, $X_1 = L(2,8) \cap L(3,35) \cap \cdots \cap L(88,100)$; i.e., the incenter is the point of concurrence of nine lines, the first of which passes through X_2 (the centroid) and X_8 (the Nagel point), the second of which passes through the circumcenter and X_{35}, and so on.

For each center $X_i = l : m : n$ in Table 1, L_i denotes the line $l\alpha + m\beta + n\gamma = 0$, called in **[19]** the *trilinear polar of* X_i^{-1}. To construct L_i, let $Q_A = AX_i^{-1} \cap BC, Q_B = BX_i^{-1} \cap CA, Q_C = CX_i^{-1} \cap AB$. Then Q_A, Q_B, Q_C are collinear on L_i **[22**, p. 150**]**. To see that this is a central line, note that L_i contains the centers $X_{100} \cdot X_i^{-1}$ and $X_{100} \cdot X_6 \cdot X_i^{-1}$, where the operation $\cdot$ is as defined at X_1.

A line $L(i, j)$ contains centers other than X_i and X_j. For example, $L(1,2)$ contains not only X_1 and X_2, but also $X_8, X_{10}, X_{42}, X_{43}, X_{78}$, and infinitely many other centers. A list of *all* such collinearities involving the 101 centers in Table 1 is available. See the final paragraph of Section 3.

TABLE 1

$\mathbf{X_1}$ **INCENTER** $\boldsymbol{\alpha} = 1$ **[5, p. 622]**, the point of concurrence of the interior angle bisectors of ABC. The incenter is the point inside ABC whose distances from the sidelines are equal. This common distance is the radius of the incircle, so named because it is inscribed in ABC. As a function of a, b, c, the inradius is given by

$$r = \frac{\sqrt{(b+c-a)(c+a-b)(a+b-c)}}{2\sqrt{a+b+c}},$$

which is the area of ABC divided by half the perimeter of ABC. Altshiller-Court [**1**, pp. 72–93] and Johnson [**22**, pp. 182–194] discuss the geometry of the incenter, the three excenters, and the four associated circles. (See FIGURE 9.) Concerning algebraic properties, X_1 is the identity of the group $(\mathbb{G}, \cdot)$, where $\mathbb{G}$ is the set of centers and $\cdot$ is defined by $(\alpha_1 : \beta_1 : \gamma_1) \cdot (\alpha_2 : \beta_2 : \gamma_2) = \alpha_1\alpha_2 \colon \beta_1\beta_2 \colon \gamma_1\gamma_2$. $X_1 = L(2,8) \cap L(3,35) \cap L(4,33) \cap L(5,11) \cap L(6,9) \cap L(7,20) \cap L(19,28) \cap L(21,31) \cap L(29,92) \cap L(30,79) \cap L(41,101) \cap L(75,86) \cap L(88,100)$. $L_1 \| L_9, L_{37}, L_{44}, L_{45}, L_{72}$. $L_1 \perp L(1,3), L(4,8), L(5,10)$.

$\mathbf{X_2}$ **CENTROID** $\boldsymbol{\alpha} = 1/a$ **[5, p. 622]**, the point of concurrence of the medians of ABC. Among its many properties is the fact that the triangles BXC, CXA, AXB have equal areas if, and only if, $X = X_2$. A theorem in high school texts is that the centroid is situated one-third of the way from each vertex to the midpoint of the opposite side. More generally, if $\mathscr{L}$ is *any* line in the plane of ABC, then $d = \frac{1}{3}(d_A + d_B + d_C)$, where d, d_A, d_B, d_C denote the distances from X_2, A, B, C to $\mathscr{L}$, respectively.

Physically speaking, X_2 is the balance point of a triangular sheet of constant thickness and constant mass density. That is, the sheet will stay balanced atop a pin head located directly under X_2. It will also stay balanced atop any knife-edge that passes directly under X_2. Unlike most centers, the centroid can be easily defined for other shapes so as to retain many of its ABC-related properties. $X_2 = L(1,8) \cap L(3,4) \cap L(6,69) \cap L(7,9) \cap L(11,55) \cap L(12,56) \cap L(13,16) \cap L(14,15) \cap L(17,62) \cap L(18,61) \cap L(32,83) \cap L(37,75) \cap L(39,76) \cap L(44,89) \cap L(45,88) \cap L(54,68) \cap L(95,97)$. $L_2 \| L_{69}, L_{81}, L_{86}$. $L_2 \perp L(2,51), L(3,6), L(4,69), L(66,68)$. L_2 is known as the **Lemoine axis.**

$\mathbf{X_3}$ **CIRCUMCENTER** $\boldsymbol{\alpha} = \cos A$, **or** $\boldsymbol{\alpha} = a(b^2 + c^2 - a^2)$ **[5, p. 623]**, the point of concurrence of the perpendicular bisectors of the sides of ABC. The distances $|AX|, |BX|, |CX|$ are equal if, and only if, $X = X_3$. This common distance is the radius of the circumcircle, which passes through A, B, and C. As a function of a, b, c, the circumradius is given by

$$R = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}.$$

The distance from the circumcenter to the incenter is $\sqrt{R(R-2r)}$, where r denotes the inradius. $X_3 = L(1,35) \cap L(2,4) \cap L(6,15) \cap L(8,100) \cap L(9,84) \cap L(13,17) \cap L(14,18) \cap L(48,71) \cap L(54,97) \cap L(63,72) \cap L(76,98)$. $L_3 \| L_{15}, L_{16}, L_{32}, L_{39}, L_{50}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_3 \perp L(1, 79), L(2, 3), L(11, 36), L(12, 35), L(13, 15), L(14, 16), L(64, 68)$. $L(3, 6)$ is known as the **Brocard diameter.**

$\mathbf{X_4}$ **ORTHOCENTER** $\boldsymbol{\alpha} = \sec A$ **[5, p. 622]**, the point of concurrence of the altitudes of ABC. The orthocenter and the vertices A, B, C provide the prototype of an *orthocentric system*, defined as a set of four points, one of which is the orthocenter of the triangle of the other three. For theorems on orthocentric systems, see [1, pp. 109–114] and [22, pp. 165–172]. The incenter and excenters form an orthocentric system, since X_1 is the orthocenter of the excentral triangle. $X_4 = X_3^{-1} = L(1,33) \cap L(2,3) \cap L(6,53) \cap L(8,72) \cap L(9,10) \cap L(11,56) \cap L(12,55) \cap L(13,61) \cap L(14,62) \cap L(15,17) \cap L(16,18) \cap L(32,98) \cap L(46,90) \cap L(52,68) \cap L(57,84) \cap L(69,76)$. $L_4 \| L_{53}$. $L_4 \perp L(3,64), L(4,51)$.

X_5 NINE-POINT CENTER $\alpha = \cos(B - C)$, or $\alpha = \cos A + 2\cos B \cos C$, or $\alpha = bc[a^2b^2 + a^2c^2 + (b^2 - c^2)^2]$ [5, p. 624], the center of the nine-point circle. Euler showed in 1765 that this circle passes through the midpoints of the sides of ABC and the feet of the altitudes of ABC, hence six of the nine points. The other three are the midpoints of AX_4, BX_4, CX_4. The radius of the nine-point circle is $R/2$, where R denotes the radius of the circumcircle [10], [35]. $X_5 = L(1,11) \cap L(2,3) \cap L(6,68) \cap L(13,18) \cap L(14,17) \cap L(49,54) \cap L(51,52) \cap L(83,98)$. $L_5 \| L_{68}$. $L_5 \perp L(3,49), L(4,52)$.

X_6 SYMMEDIAN POINT (LEMOINE POINT) $\alpha = a$ [19, p. 86], the point of concurrence of the symmedians (obtained by reflecting the medians about the corresponding interior angle bisectors). X_6 is the point (α, β, γ), given here in actual trilinear distances, for which $\alpha^2 + \beta^2 + \gamma^2$ achieves its minimum value. A center X is the centroid of its own pedal triangle if, and only if, $X = X_6$. Moreover, of all triangles inscribed in ABC, that for which the sum of areas of squares of the side-lengths is minimal is the pedal triangle of X_6 [22, p. 217]. $X_6 = X_2^{-1} = L(1,9) \cap L(2,69) \cap L(3,15) \cap L(4,53) \cap L(5,68) \cap L(13,14) \cap L(17,18) \cap L(19,34) \cap L(24,54) \cap L(25,51) \cap L(31,42) \cap L(41,48) \cap L(43,87) \cap L(76,83) \cap L(88,89)$. L_6 is the line at infinity; it is parallel to and perpendicular to every central line.

X_7 GERGONNE POINT $\alpha = 1/a(b + c - a)$, or $\alpha = \sec^2 A/2$ [19, p. 22]. Let A', B', C' denote the points in which the incircle meets the sides BC, CA, AB, respectively. The lines AA', BB', CC' concur in X_7 [1, pp. 160–164], [15, p. 83]. $X_7 = L(1,20) \cap L(2,9) \cap L(8,65) \cap L(21,56) \cap L(27,81)$. $L_7 \perp L(3,101)$.

X_8 NAGEL POINT $\alpha = (b + c - a)/a$, or $\alpha = \csc^2 A/2$ [19, p. 20]. Let A', B', C' denote the points in which the A-excircle meets BC, the B-excircle meets CA, and the C-excircle meets AB. The lines AA', BB', CC' concur in X_8. Another construction of A' is to start at A and trace around ABC half its perimeter. The constructions of B' and C' are carried out similarly, of course [1, pp. 160–164], [15, p. 83]. $X_8 = L(1,2) \cap L(3,100) \cap L(4,72) \cap L(7,65) \cap L(20,40) \cap L(21,55)$. $L_8 \perp L(40,43)$.

X_9 MITTENPUNKT $\alpha = b + c - a$, or $\alpha = \cot A/2$ [13], the symmedian point of the excentral triangle. "Mittenpunkt" is the name given by Nagel in 1836; it is sometimes translated as "middlespoint" [13], [19, p. 90]. $X_9 = L(1,6) \cap L(2,7) \cap L(3,84) \cap L(4,10) \cap L(21,41) \cap L(35,90) \cap L(46,79) \cap L(48,101)$. $L_9 \| L_1, L_{37}, L_{44}, L_{45}, L_{72}$. $L_9 \perp L(1,3), L(4,8), L(5,10)$.

X_{10} SPIEKER CENTER $\alpha = (b + c)/a$ [6, p. 81]. The Spieker circle is the incircle of the medial triangle. Its center, X_{10}, is the center of mass of the perimeter of ABC. (In a bad moment, you might have guessed this center of mass to be X_2.) [22, pp. 226–229, 249] $X_{10} = L(1,2) \cap L(4,9) \cap L(12,65) \cap L(21,35) \cap L(46,63) \cap L(75,76) \cap L(82,83) \cap L(98,101)$. $L_{10} \perp L(3,31)$.

X_{11} **FEUERBACH POINT** $\alpha = 1 - \cos(B - C)$ [39, p. 127], the point of intersection of the nine-point circle and the incircle. Feuerbach's famous theorem states that the nine-point circle is tangent to the incircle and the three excircles. Johnson [22, p. 200] describes this as "perhaps the most famous of all theorems of the triangle, aside from those known in ancient times." In an algebraic context [8], Feuerbach's theorem is called "one of the most fascinating facts in euclidean geometry" [21]. $X_{11} = L(1,5) \cap L(2,55) \cap L(4,56)$.

X_{12} **HARMONIC CONJUGATE OF** X_{11} **with respect to** X_1 **and** X_5 $\alpha = 1 + \cos(B - C)$ [21]. Let A', B', C' be the touch points of the nine-point circle with the A-excircle, B-excircle, and C-excircle, respectively. The point of concurrence of AA', BB', CC' is X_{12}. $X_{12} = L(1,5) \cap L(2,56) \cap L(4,55) \cap L(10,65)$.

X_{13} **1st ISOGONIC CENTER (FERMAT POINT)** $\alpha = \csc(A + \pi/3)$ [19, p. 107], or

$$\alpha = bc\left[\left(c^2a^2 + (c^2 + a^2 - b^2)^2\right)\right]\left[\left(a^2b^2 - (a^2 + b^2 - c^2)^2\right)\right] \times\left[\left(4\sqrt{\mathscr{P}} - \sqrt{3}\,(b^2 + c^2 - a^2)\right)\right].$$

The latter shows that X_{13} is a regular center. Construct the equilateral triangle $BA'C$ having base BC and vertex A' on the side of BC opposite that of vertex A; similarly construct equilateral triangles $CB'A$ and $AC'B$ based on the other two sides. The lines AA', BB', CC' concur in X_{13}. If no angle of ABC exceeds $2\pi/3$, then X_{13} is the only center X for which the angles BXC, CXA, AXB are equal. X_{13} also minimizes the sum $|AX| + |BX| + |CX|$. The antipedal triangle of X_{13} is equilateral and has area $2\sqrt{\mathscr{P}}(1 + \cot\omega\cot\pi/3)$, where ω denotes the Brocard angle. See FIGURE 11. $X_{13} = L(2,16) \cap L(3,17) \cap L(4,61) \cap L(5,18) \cap L(6,14) \cap L(15,30)$. $L_{13} \| L_{14}$. $L_{13} \perp L(3,74), L(4,94)$.

X_{14} **2nd ISOGONIC CENTER** $\alpha = \csc(A - \pi/3)$ [19, p. 107]. Construct the equilateral triangle $BA'C$ on BC with A' on the side of BC that contains vertex A; similarly construct equilateral triangles $CB'A$ and $AC'B$ on the other two sides. The point of concurrence of the lines AA', BB', CC' is X_{14}. The antipedal triangle of X_{14} is equilateral and has area $2\sqrt{\mathscr{P}}(-1 + \cot\omega\cot\pi/3)$. $X_{14} = L(2,15) \cap L(3,18) \cap L(4,62) \cap L(5,17) \cap L(6,13)$. $L_{14} \| L_{13}$. $L_{14} \perp L(3,74), L(4,94)$.

X_{15} **1st ISODYNAMIC POINT** $\alpha = \sin(A + \pi/3)$ [19, p. 106]. Let U and V be the points on sideline BC met by the interior and exterior bisectors of angle A. The circle having diameter UV is the A-Apollonian circle. B- and C-Apollonian circles are similarly constructed. These circles pass through the respective vertices A, B, C and also through the two isodynamic points. The pedal triangle of X_{15} is equilateral and has area $\frac{1}{2}\sqrt{\mathscr{P}}(1 + \cot\omega\cot\pi/3)$. See FIGURE 11. $X_{15} = X_{13}^{-1} = L(2,14) \cap L(3,6) \cap L(4,17) \cap L(13,30)$. $L_{15} \| L_3, L_{16}, L_{32}, L_{39}, L_{50}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_{15} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{16} **2nd ISODYNAMIC POINT** $\alpha = \sin(A - \pi/3)$ [19, p. 106]. Another construction of the isodynamic points is given by Rigby [38]. The pedal triangle of X_{16} is equilateral and has area $\frac{1}{2}\sqrt{\mathscr{P}}(-1 + \cot\omega\cot\pi/3)$. $X_{16} = X_{14}^{-1} = L(2,13) \cap L(3,6) \cap L(4,18)$. $L_{16} \| L_3, L_{15}, L_{32}, L_{39}, L_{50}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_{16} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{17} **1st NAPOLEON POINT** $\alpha = \csc(A + \pi/6)$ [6, pp. 442–444]. Let X, Y, Z be the centers of the equilateral triangles in the construction of X_{13}. The lines AX, BY, CZ concur in X_{17}. Questionably attributed to Napoleon Bonaparte is the theorem that the triangle XYZ is equilateral [38]; the Napoleon points are so named because of their connection with the theorem. $X_{17} = L(2,62) \cap L(3,13) \cap L(4,15) \cap L(5,14) \cap L(6,18)$. $L_{17} \| L_{18}$. $L_{17} \perp L(3,54), L(4,93), L(5,51)$.

X_{18} 2nd NAPOLEON POINT $\alpha = \csc(A - \pi/6)$ [6, pp. 442–444]. Let X, Y, Z be the centers of the equilateral triangles in the construction of X_{14}. The lines AX, BY, CZ concur in X_{18}. $X_{18} = L(2,61) \cap L(3,14) \cap L(4,16) \cap L(5,13) \cap L(6,17)$. $L_{18} \| L_{17}$. $L_{18} \perp L(3,54), L(4,93), L(5,51)$.

X_{19} CRUCIAL POINT $\alpha = \tan A$, or $\alpha = \sin 2B + \sin 2C - \sin 2A$ [34], the homothetic center of the orthic triangle and the triangular hull of the three excircles (the triangle whose sides are those external common tangents of the three pairs of excircles of ABC that are not sides of ABC). See FIGURE 12. $X_{19} = L(1,28) \cap L(4,9) \cap L(6,34) \cap L(25,33) \cap L(27,63)$. $L_{19} \| L_{34}, L_{65}$. $L_{19} \perp L(1,84), L(4,65), L(40,64)$.

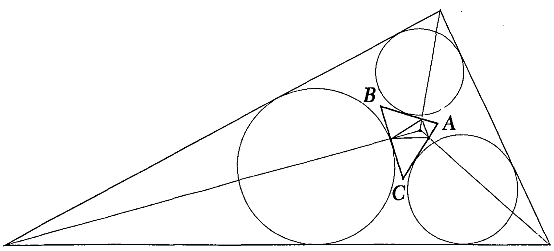

FIGURE 12

X_{19}, the homothetic center of the orthic triangle and the triangular hull of the 3 excircles. (The name "crucial point" derives from the name of the publication in which the point first appeared.)

X_{20} DE LONGCHAMPS POINT $\alpha = \cos A - \cos B \cos C$, the reflection of the orthocenter about the circumcenter. X_{20} is also the orthocenter of the anticomplementary triangle [3], [41]. $X_{20} = L(1,7) \cap L(2,3) \cap L(8,40) \cap L(64,69) \cap L(68,74)$.

X_{21} SCHIFFLER POINT $\alpha = 1/(\cos B + \cos C)$ [47]. The Euler lines of the four triangles X_1BC, X_1CA, X_1AB, and ABC concur in X_{21} [40]. $X_{21} = L(1,31) \cap L(2,3) \cap L(7,56) \cap L(8,55) \cap L(9,41) \cap L(10,35) \cap L(36,79)$.

X_{22} EXETER POINT $\alpha = a(b^4 + c^4 - a^4)$ [31]. Let A' be the point other than A where the median through A meets the circumcircle of ABC; define B' and C' similarly. Then X_{22} is the center of perspective of the triangle $A'B'C'$ and the tangential triangle. The point and its position on the Euler line were first noticed during the 1986 summer computer-and-mathematics conference at Phillips Exeter Academy—hence the name, Exeter point. $L_{22} \perp L(4,83)$.

X_{23} FAR-OUT POINT $\alpha = a(b^4 + c^4 - a^4 - b^2c^2)$ [47]. Let Γ be the circumcircle and m_A, m_B, m_C the medians of ABC. Let DEF be the anticomplementary triangle of ABC. Let Γ' be the circumcircle of DEF. Let $A'B'C'$ be the triangle formed by the tangents to Γ at the points other than A, B, C where m_A, m_B, m_C meet Γ. Finally, let A'', B'', C'' be the points other than D, E, F where m_A, m_B, m_C meet Γ'. Then X_{23} is the center of perspective of triangles $A'B'C'$ and $A''B''C''$. The author conjectured and Peter Yff [47] proved that as $a:b:c$ approaches $1:1:1$, this point scoots along the Euler line to infinity, hence its name [30]. $X_{23} = L(2,3) \cap L(94,98)$.

X_{24} CENTER OF PERSPECTIVE OF *ABC* and ORTHIC-OF-ORTHIC TRIANGLE $\alpha = \sec A \cos 2A$ **[6, p. 85].** $X_{24} = L(2,3) \cap L(6,54) \cap L(33,35) \cap L(34,36) \cap L(64,74)$. $L_{24} \| L_{54}$. $L_{24} \perp L(4,54)$.

X_{25} HOMOTHETIC CENTER OF ORTHIC and TANGENTIAL TRIANGLES $\alpha = \sin A \tan A$ **[6, p. 85]**, a point on the Euler line [1, p. 98]. $X_{25} = L(2,3) \cap L(6,51) \cap L(19,33) \cap L(34,56) \cap L(41,42)$. $L_{25} \| L_{51}$. $L_{25} \perp L(3,66), L(4,6), L(20,64), L(67,74)$.

X_{26} CIRCUMCENTER OF THE TANGENTIAL TRIANGLE $\alpha = a(b^2 \cos 2B + c^2 \cos 2C - a^2 \cos 2A)$, a point on the Euler line. See [1, p. 104]; trilinears are not given there, however, and perhaps appear here for the first time.

X_{27} $\alpha = (\sec A)/(b+c)$, a point on the Euler line. $X_{27} = L(2,3) \cap L(7,81) \cap L(19,63)$.

X_{28} $\alpha = (\tan A)/(b+c)$, a point on the Euler line. $X_{28} = L(1,19) \cap L(2,3) \cap L(34,57) \cap L(60,81)$.

X_{29} $\alpha = (b+c-a)(\sec A)/(b+c)$, a point on the Euler line. $X_{29} = L(1,92) \cap L(2,3) \cap L(33,78) \cap L(34,77)$.

X_{30} $\alpha = \cos A - 2\cos B \cos C$, the point of intersection of the Euler line and the line at infinity. To say that X_{30} lies on distinct lines is to say that those lines are parallel. For proofs of some of the following parallelisms, see [12]. $X_{30} = L(1,79) \cap L(2,3) \cap L(11,36) \cap L(12,35) \cap L(13,15) \cap L(14,16) \cap L(64,68)$.

X_{31} 2nd POWER POINT $\alpha = a^2$ **[20], [23].** $X_{31} = L(1,21) \cap L(6,42) \cap L(32,41) \cap L(43,100) \cap L(75,82)$. $L_{31} \| L_{42}, L_{55}, L_{71}$. $L_{31} \perp L(1,7), L(4,9)$.

X_{32} 3rd POWER POINT $\alpha = a^3$ **[20], [23].** $X_{32} = L(2,83) \cap L(3,6) \cap L(4,98) \cap L(31,41)$. $L_{32} \| L_3, L_{15}, L_{16}, L_{39}, L_{50}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_{32} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{33} $\alpha = 1 + \sec A$, the center of perspective of the orthic triangle and the triangle of the alternate interior tangents, defined at X_{55}. $X_{33} = L(1,4) \cap L(19,25) \cap L(24,35) \cap L(29,78) \cap L(47,90) \cap L(64,65)$. $L_{33} \perp L(1,64)$.

X_{34} $\alpha = 1 - \sec A$, the center of perspective of the orthic triangle and the reflection about X_1 of the triangle of the alternate interior tangents, defined at X_{55}. $X_{34} = L(1,4) \cap L(6,19) \cap L(24,36) \cap L(25,56) \cap L(28,57) \cap L(29,77) \cap L(46,47)$. $L_{34} \| L_{65}$. $L_{34} \perp L(1,84), L(4,65), L(40,64)$.

X_{35} $\alpha = 1 + 2\cos A$. $X_{35} = L(1,3) \cap L(9,90) \cap L(10,21) \cap L(24,33) \cap L(42,58) \cap L(73,74)$.

X_{36} $\alpha = 1 - 2\cos A$. $X_{36} = L(1,3) \cap L(21,79) \cap L(24,34) \cap L(54,73) \cap L(58,60) \cap L(84,90)$.

X_{37} $\alpha = b + c$. Algebraically speaking, X_{37} is perhaps the simplest unnamed center. $X_{37} = L(1,6) \cap L(2,75) \cap L(19,25) \cap L(65,71)$. $L_{37} \| L_{44}, L_{45}, L_{72}$. $L_{37} \perp L(1,3), L(4,8), L(5,10)$.

X_{38} $\alpha = b^2 + c^2$, collinear with centers X_1, X_{21}.

X_{39} BROCARD MIDPOINT $\alpha = a(b^2 + c^2)$, or $\alpha = \sin(A + \omega)$, where ω is the Brocard angle. This is the midpoint of the two Brocard points, described near the end of Section 2. The trilinears of X_{39} are easily found by averaging the actual trilinear distances of the two Brocard points. $X_{39} = L(2,76) \cap L(3,6) \cap L(83,99)$. $L_{39} \| L_3, L_{15}, L_{16}, L_{32}, L_{50}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_{39} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{40} $\alpha = \cos B + \cos C - \cos A - 1$ [47], the point of concurrence of the perpendiculars from the excenters to the respective sides. $X_{40} = L(1,3) \cap L(4,9) \cap L(8,20) \cap L(64,72) \cap L(78,100) \cap L(80,90)$.

X_{41} $\alpha = a^2(b + c - a)$. $X_{41} = L(1,101) \cap L(6,48) \cap L(9,21) \cap L(25,42) \cap L(31,32)$. $L_{41} \| L_{48}, L_{56}, L_{73}$. $L_{41} \perp L(1,4), L(3,10), L(8,20), L(36,80)$.

X_{42} $\alpha = a(b + c)$. $X_{42} = L(1,2) \cap L(6,31) \cap L(25,41) \cap L(35,58) \cap L(65,73) \cap L(81,100)$. $L_{42} \| L_{55}, L_{71}$. $L_{42} \perp L(1,7), L(4,9)$.

X_{43} $\alpha = ca + ab - bc$. $X_{43} = L(1,2) \cap L(6,87) \cap L(31,100)$. $L_{43} \| L_{87}$.

X_{44} $\alpha = b + c - 2a$. $X_{44} = L(1,6) \cap L(2,89)$. $L_{44} \| L_{37}, L_{45}, L_{72}$. $L_{44} \perp L(1,3), L(4,8), L(5,10)$.

X_{45} $\alpha = 2b + 2c - a$. $X_{45} = L(1,6) \cap L(2,88)$. $L_{45} \| L_{37}, L_{44}, L_{72}$. $L_{45} \perp L(1,3), L(4,8), L(5,10)$.

X_{46} $\alpha = \cos B + \cos C - \cos A$. $X_{46} = L(1,3) \cap L(4,90) \cap L(9,79) \cap L(10,63) \cap L(34,47) \cap L(80,84)$.

X_{47} $\alpha = \cos 2A$. $X_{47} = L(1,21) \cap L(33,90) \cap L(34,46) \cap L(91,92)$.

X_{48} $\alpha = \sin 2A$. $X_{48} = L(1,19) \cap L(3,71) \cap L(6,41) \cap L(9,101)$. $L_{48} \| L_{56}, L_{73}$. $L_{48} \perp L(1,4), L(3,10), L(8,20), L(36,80)$.

X_{49} $\alpha = \cos 3A$. $X_{49} = L(5,54) \cap L(93,94)$. $L_{49} \perp L(3,70), L(26,68)$.

X_{50} $\alpha = \sin 3A$. X_{50} is collinear with centers X_3 and X_6. $L_{50} \| L_3, L_{15}, L_{16}, L_{32}, L_{39}, L_{52}, L_{58}, L_{61}, L_{62}$. $L_{50} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{51} CENTROID OF THE ORTHIC TRIANGLE $\alpha = a^2 \cos(B - C)$ [6, p. 92]. $X_{51} = L(5,52) \cap L(6,25)$. $L_{51} \| L_{25}$. $L_{51} \perp L(3,66), L(4,6), L(20,64), L(67,74)$.

X_{52} ORTHOCENTER OF THE ORTHIC TRIANGLE $\alpha = \cos 2A \cos(B - C)$ [6, p. 92]. $X_{52} = L(3,6) \cap L(4,68) \cap L(5,51)$. $L_{52} \| L_3, L_{15}, L_{16}, L_{32}, L_{39}, L_{50}, L_{58}, L_{61}, L_{62}$. $L_{52} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

X_{53} SYMMEDIAN POINT OF THE ORTHIC TRIANGLE $\alpha = \tan A \cos(B - C)$ [6, p. 92], collinear with centers X_4 and X_6. $L_{53} \perp L(3,64), L(4,51)$.

X_{54} $\alpha = \sec(B - C)$. $X_{54} = X_5^{-1} = L(2,68) \cap L(3,97) \cap L(5,49) \cap L(6,24) \cap L(36,73) \cap L(69,95)$. $L_{54} \perp L(4,54)$.

X_{55} $\alpha = a(b + c - a)$, or $\alpha = \cos^2(A/2)$, or $\alpha = 1 + \cos A$. This is the center of perspective of triangles $A'B'C'$ and $A''B''C''$ defined as follows: $A'B'C'$ is the tangential triangle of ABC; reflect line BC about the internal bisector of angle A, and line CA about the internal bisector of angle B, and line AB about the internal bisector of angle C to form $A''B''C''$. (Note that this reflection of BC, like BC itself, is internally tangent to both the incircle and the A-excircle of ABC, and similarly for CA and AB; accordingly, we call $A''B''C''$ the *triangle of the alternate interior tangents*.)

George Berzsenyi has noted that there are three congruent circles contained within ABC: Γ_A, internally tangent to both AB and AC, Γ_B, internally tangent to both BC and BA, and Γ_C, internally tangent to both CA and CB, which meet in a single point X. (See FIGURE 13.) To determine whether X is a known center, Peter Yff [47] found its trilinears, thereby discovering that $X = X_{55}$. He also found the radius of the three circles to be

$$\frac{a \sin B \sin C}{2 \sin A \sin B \sin C + \sin A + \sin B + \sin C}.$$

In a letter, Eric Antokoletz pointed out that this radius equals $Rr/(R+r)$, where R and r denote the radii of the circumcircle and the incircle, respectively. Antokoletz noticed that if the words "contained within ABC" are suppressed, then there are three more such circles, concurrent in X_{55}, all with radius $Rr/(R-r)$. $X_{55} = X_7^{-1} = L(1,3) \cap L(2,11) \cap L(4,12) \cap L(6,31) \cap L(8,21) \cap L(19,25) \cap L(64,73)$. $L_{55} \| L_{42}, L_{71}$. $L_{55} \perp L(1,7), L(4,9)$.

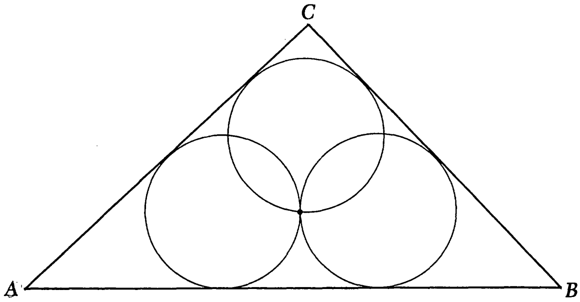

FIGURE 13
The center X_{55}.

$\mathbf{X_{56}}$ $\boldsymbol{\alpha = \sin^2(A/2)}$, **or** $\boldsymbol{\alpha = -1 + \cos A}$. This is the center of perspective of the tangential triangle and the reflection of the triangle of the alternate interior tangents about X_1. It is the harmonic conjugate of X_{55} with respect to X_1 and X_3. $X_{56} = X_8^{-1} = L(1,3) \cap L(2,12) \cap L(4,11) \cap L(6,41) \cap L(7,21) \cap L(25,34)$. $L_{56} \| L_{48}, L_{73}$. $L_{56} \perp L(1,4), L(3,10), L(8,20), L(36,80)$.

$\mathbf{X_{57}}$ $\boldsymbol{\alpha = 1/(b+c-a)}$, **or** $\boldsymbol{\alpha = \tan A/2}$. $X_{57} = X_9^{-1} = L(1,3) \cap L(2,7) \cap L(4,84) \cap L(28,34) \cap L(77,81) \cap L(79,90)$. $L_{57} \perp L(3,9), L(4,7), L(20,72)$.

$\mathbf{X_{58}}$ $\boldsymbol{\alpha = a/(b+c)}$. $X_{58} = X_{10}^{-1} = L(1,21) \cap L(3,6) \cap L(28,34) \cap L(35,42) \cap L(36,60)$. $L_{58} \| L_3, L_{15}, L_{16}, L_{32}, L_{39}, L_{50}, L_{52}, L_{61}, L_{62}$. $L_{58} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

$\mathbf{X_{59}}$ $\boldsymbol{\alpha = 1/(1-\cos(B-C))}$. $X_{59} = X_{11}^{-1}$.

$\mathbf{X_{60}}$ $\boldsymbol{\alpha = 1/(1+\cos(B-C))}$. $X_{60} = X_{12}^{-1} = L(28,81) \cap L(36,58)$.

$\mathbf{X_{61}}$ $\boldsymbol{\alpha = \sin(A + \pi/6)}$. $X_{61} = X_{17}^{-1} = L(2,18) \cap L(3,6) \cap L(4,13) \cap L(5,14)$. $L_{61} \| L_3, L_{15}, L_{16}, L_{32}, L_{39}, L_{50}, L_{52}, L_{58}, L_{62}$. $L_{61} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

$\mathbf{X_{62}}$ $\boldsymbol{\alpha = \sin(A - \pi/6)}$. $X_{62} = X_{18}^{-1} = L(2,17) \cap L(3,6) \cap L(4,14) \cap L(5,13)$. $L_{62} \| L_3, L_{15}, L_{16}, L_{32}, L_{39}, L_{50}, L_{52}, L_{58}, L_{61}$. $L_{62} \perp L(1,79), L(2,3), L(11,36), L(12,35), L(13,15), L(14,16), L(64,68)$.

$\mathbf{X_{63}}$ $\boldsymbol{\alpha = \cot A}$, **or** $\boldsymbol{\alpha = b^2 + c^2 - a^2}$. $X_{63} = X_{19}^{-1} = L(1,21) \cap L(2,7) \cap L(3,72) \cap L(8,20) \cap L(10,46) \cap L(19,27) \cap L(69,71)$. $L_{63} \perp L(3,37), L(4,75)$.

X_{64} $\alpha = 1/(\cos A - \cos B \cos C)$. $X_{64} = X_{20}^{-1} = L(20,69) \cap L(24,74) \cap L(33,65) \cap L(40,72) \cap L(55,73)$. $L_{64} \perp L(4,64)$.

X_{65} $\alpha = \cos B + \cos C$. $X_{65} = X_{21}^{-1} = L(1,3) \cap L(6,19) \cap L(7,8) \cap L(10,12) \cap L(33,64) \cap L(37,71) \cap L(42,73) \cap L(68,91) \cap L(79,80)$. $X_{65} \perp L(1,84), L(4,65), L(40,64)$.

X_{66} $\alpha = 1/a(b^4 - c^4 - a^4)$. $X_{66} = X_{22}^{-1}$. $L_{66} \perp L(4,66), L(6,64)$.

X_{67} $\alpha = 1/a(b^4 + c^4 - a^4 - b^2c^2)$. $X_{67} = X_{23}^{-1}$. $L_{67} \perp L(4,67), L(6,74)$.

X_{68} $\alpha = \cos A \sec 2A$. $X_{68} = X_{24}^{-1} = L(2,54) \cap L(4,52) \cap L(5,6) \cap L(20,74) \cap L(65,91)$. $L_{68} \perp L(3,49), L(4,52)$.

X_{69} $\alpha = \cos A \csc^2 A$. $X_{69} = X_{25}^{-1} = L(2,6) \cap L(4,76) \cap L(7,8) \cap L(20,64) \cap L(54,95) \cap L(63,71) \cap L(73,77) \cap L(74,99)$. $L_{69} \| L_{81}, L_{86}$. $L_{69} \perp L(2,51), L(3,6), L(4,69), L(66,68)$.

X_{70} $\alpha = 1/a(b^2\cos 2B + c^2\cos 2C - a^2\cos 2A)$. $X_{70} = X_{26}^{-1}$. $L_{70} \perp L(4,70)$.

X_{71} $\alpha = (b+c)\cos A$. $X_{71} = X_{27}^{-1} = L(3,48) \cap L(4,9) \cap L(6,31) \cap L(37,65) \cap L(63,69) \cap L(74,101)$. $L_{71} \perp L(1,7), L(4,9)$.

X_{72} $\alpha = (b+c)\cot A$. $X_{72} = X_{28}^{-1} = L(1,6) \cap L(3,63) \cap L(4,8) \cap L(10,12) \cap L(40,64) \cap (74,100)$. $L_{72} \| L_{37}, L_{44}, L_{45}$. $L_{72} \perp L(1,3), L(4,8), L(5,10)$.

X_{73} $\alpha = ((b+c)\cos A)/(b+c-a)$. $X_{73} = X_{29}^{-1} = L(1,4) \cap L(6,41) \cap L(35,74) \cap L(36,54) \cap L(42,65) \cap L(55,64) \cap L(69,77)$. $L_{73} \| L_{48}, L_{56}$. $L_{73} \perp L(1,4), L(3,10), L(8,20), L(36,80)$.

X_{74} $\alpha = 1/\cos A - 2\cos B \cos C)$. As the isogonal conjugate of the point in which the Euler line meets the line at infinity, X_{74} lies on the circumcircle. $X_{74} = X_{30}^{-1} = L(20,68) \cap L(24,64) \cap L(35,73) \cap L(69,99) \cap L(71,101) \cap L(72,100)$. $L_{74} \perp L(4,74)$.

X_{75} $\alpha = a^{-2}$. $X_{75} = X_{31}^{-1} = L(1,86) \cap L(2,37) \cap L(7,8) \cap L(10,76) \cap L(19,27) \cap L(31,82)$.

X_{76} **3rd BROCARD POINT** $\alpha = a^{-3}$ [6, p. 66]. $X_{76} = X_{32}^{-1} = L(2,39) \cap L(3,98) \cap L(4,69) \cap L(6,83) \cap L(10,75) \cap L(95,96)$. $L_{76} \| L_{83}$.

X_{77} $\alpha = 1/(1 + \sec A)$. $X_{77} = X_{33}^{-1} = L(1,7) \cap L(29,34) \cap L(57,81) \cap L(69,73)$. $L_{77} \| L(4,85)$.

X_{78} $\alpha = 1/(1 - \sec A)$. $X_{78} = X_{34}^{-1} = L(1,2) \cap L(3,63) \cap L(9,21) \cap L(29,33) \cap L(40,100) \cap L(69,73)$.

X_{79} $\alpha = 1/(1 + 2\cos A)$. $X_{79} = X_{35}^{-1} = L(1,30) \cap L(9,46) \cap L(21,36) \cap L(57,90) \cap L(65,80)$.

X_{80} $\alpha = 1/(1 - 2\cos A)$. $X_{80} = X_{36}^{-1} = L(1,5) \cap L(10,21) \cap L(40,90) \cap L(46,84) \cap L(65,79)$.

X_{81} $\alpha = 1/(b+c)$. $X_{81} = X_{37}^{-1} = L(1,21) \cap L(2,6) \cap L(7,27) \cap L(28,60) \cap L(42,100) \cap L(57,77)$. $L_{81} \| L_{69}, L_{86}$. $L_{81} \perp L(2,51), L(3,6), L(4,69), L(66,68)$.

X_{82} $\alpha = 1/(b^2 + c^2)$. $X_{82} = X_{38}^{-1} = L(10,83) \cap L(31,75)$.

X_{83} $\alpha = 1/a(b^2 + c^2)$. $X_{83} = X_{39}^{-1} = L(2,32) \cap L(5,98) \cap L(6,76) \cap L(10,82) \cap L(39,99)$.

X_{84} $\alpha = 1/(\cos B + \cos C - \cos A - 1)$ [32] Let A', B', C' be the excenters of ABC. The perpendiculars from B' to AB and from C' to AC meet in a point A''. Points B'' and C'' are determined analogously. X_{84} is the point of concurrence of lines

AA'', BB'', CC''. X_{84} is also the reflection of X_1 about X_3. $X_{84} = X_{40}^{-1} = L(3,9) \cap L(4,57) \cap L(8,20) \cap L(36,90) \cap L(46,80)$.

$\mathbf{X_{85}}$ $\boldsymbol{\alpha = 1/a^2(b+c-a)}$. $X_{85} = X_{41}^{-1} = L(7,8) \cap L(29,34)$.

$\mathbf{X_{86}}$ $\boldsymbol{\alpha = 1/a(b+c)}$. $X_{86} = X_{42}^{-1} = L(1,75) \cap L(2,6) \cap L(7,21) \cap L(29,34)$. $L_{86} \| L(2,51), L(3,6), L(4,69), L(66,68)$.

$\mathbf{X_{87}}$ $\boldsymbol{\alpha = 1/(ca+ab-bc)}$. $X_{87} = X_{43}^{-1}$, collinear with centers X_6 and X_{43}.

$\mathbf{X_{88}}$ $\boldsymbol{\alpha = 1/(b+c-2a)}$. $X_{88} = X_{44}^{-1} = L(1,100) \cap L(2,45) \cap L(6,89)$. $L_{88} \| L_{89}$.

$\mathbf{X_{89}}$ $\boldsymbol{\alpha = 1/(2b+2c-a)}$. $X_{89} = X_{45}^{-1} = L(2,44) \cap L(6,88)$. $L_{89} \| L_{88}$.

$\mathbf{X_{90}}$ $\boldsymbol{\alpha = 1/(\cos B + \cos C - \cos A)}$. $X_{90} = X_{46}^{-1} = L(4,46) \cap L(9,35) \cap L(33,47) \cap L(36,84) \cap L(40,80) \cap L(57,79)$.

$\mathbf{X_{91}}$ $\boldsymbol{\alpha = \sec 2A}$. $X_{91} = X_{47}^{-1} = L(47,92) \cap L(65,68)$.

$\mathbf{X_{92}}$ $\boldsymbol{\alpha = \csc 2A}$. $X_{92} = X_{48}^{-1} = L(1,29) \cap L(4,8) \cap L(19,27) \cap L(47,91)$.

$\mathbf{X_{93}}$ $\boldsymbol{\alpha = \sec 3A}$. $X_{93} = X_{49}^{-1}$, collinear with centers X_{49} and X_{94}.

$\mathbf{X_{94}}$ $\boldsymbol{\alpha = \csc 3A}$. $X_{94} = X_{50}^{-1} = L(23,98) \cap L(49,93)$.

$\mathbf{X_{95}}$ $\boldsymbol{\alpha = b^2c^2\sec(B-C)}$. $X_{95} = X_{51}^{-1} = L(2,97) \cap L(54,69) \cap L(76,96)$.

$\mathbf{X_{96}}$ $\boldsymbol{\alpha = \sec 2A \sec(B-C)}$. $X_{96} = X_{52}^{-1} = L(2,54) \cap L(76,95)$.

$\mathbf{X_{97}}$ $\boldsymbol{\alpha = \cot A \sec(B-C)}$. $X_{97} = X_{53}^{-1} = L(2,95) \cap L(3,54)$. $L_{97} \perp L(4,95)$.

$\mathbf{X_{98}}$ **TARRY POINT** $\boldsymbol{\alpha = bc/(b^4+c^4-a^2b^2-a^2c^2)}$, **or** $\boldsymbol{\alpha = \sec(A+\omega)}$, **where** $\boldsymbol{\omega}$ **is the Brocard angle [19, p. 102].** X_{98} lies on the circumcircle, diametrically opposite X_{99}. The lines $L(98,14)$ and $L(4,13)$ are parallel. $X_{98} = L(3,76) \cap L(4,32) \cap L(5,83) \cap L(10,101) \cap L(23,94)$.

$\mathbf{X_{99}}$ **STEINER POINT** $\boldsymbol{\alpha = bc(a^2-b^2)(a^2-c^2)}$ **[19, p. 102], [6, p. 66].** X_{99} lies on the circumcircle, diametrically opposite X_{98}. The lines $L(99,69)$ and $L(6,13)$ are parallel. $X_{99} = L(3,76) \cap L(39,83) \cap L(69,74)$.

$\mathbf{X_{100}}$ $\boldsymbol{\alpha = (c-a)(a-b)}$, a center on the circumcircle, perhaps the simplest of all such centers from an algebraic point of view, and certainly the simplest "2-center," as defined in [33]. X_{100} has a remarkable affinity with X_{11} concerning parallel lines:

$L(100,4) \| L(11,3)$	$L(100,20) \| L(11,4)$	$L(100,69) \| L(11,6)$
$L(100,56) \| L(11,8)$	$L(100,88) \| L(11,10)$	$L(100,12) \| L(11,21)$
$L(100,75) \| L(11,37)$	$L(100,76) \| L(11,39)$	$L(100,78) \| L(11,65)$

Other parallelisms involving X_{100} are $L(100,36) \| L(1,2)$, $L(100,3) \| L(1,5)$, $L(100,63) \| L(9,48)$. $X_{100} = L(1,88) \cap L(2,11) \cap L(3,8) \cap L(10,21) \cap L(31,43) \cap L(40,78) \cap L(42,81) \cap L(72,74)$.

$\mathbf{X_{101}}$ $\boldsymbol{\alpha = a(c-a)(a-b)}$, a center on the circumcircle. $X_{101} = L(1,41) \cap L(9,48) \cap L(10,98) \cap L(71,74)$. $L_{101} \perp L(1,5), L(3,8)$.

You may have some questions about the equations, parallelisms, and perpendicularities that come at the end of entries in Table 1. First, how were they found and have they been proved? Second, are they "comprehensive"—or are there some cases that are missing from the list? Third, can it ever happen that two of the lines L_i are perpendicular?

In answer to the first two questions, computer programs were written using trilinear coordinates for the centers. These programs looped through all possible collinearities, parallelisms, and perpendicularities. For each possibility, the relation was either found to hold with great numerical accuracy for a number of different triangles or else a counterexample was found. *All* such findings are included in Table 1 together with two Appendices that are not published here but which I will be pleased to send you on request. Most of the relations have been verified by computer algebra system, using methods of De Vogelaere in [11] and [12]. In answer to the third question, yes, surprisingly, two trilinear polars *can* be perpendicular, even though this does not occur in Table 1. This and related questions are discussed in [33].

4. More Centers

There are many other interesting centers. For some of them, a center-function $f(a, b, c)$ can be written as a nonregular but still algebraic function in a, b, c. However, it is not implicit in the definition of triangle center that this be possible or that the center be Euclidean-constructible. A thought-provoking example of a nonalgebraic (hence nonconstructible) center is $A : B : C$; that is, the point whose distances from the sidelines of ABC are proportional to the angles A, B, C. Table 2 contains some nonregular centers and a few more regular centers.

TABLE 2

Y_1 ISOPERIMETRIC POINT (CENTER OF AN INNER SODDY CIRCLE) $\alpha = 1 - 2\sqrt{\mathcal{P}}/a(b+c-a)$, or $\alpha = \sec(A/2)\cos(B/2)\cos(C/2) - 1$ [42]. This point appears in three different guises. We ask first if in the plane of ABC there is a point for which the perimeters of triangles BYC, CYA, AYB are equal. Veldkamp [42] showed that the answer is yes if, and only if, the largest angle of ABC does not exceed $2\,\mathrm{Sin}^{-1}(4/5)$, or equivalently, $a + b + c > 4R + r$, where R and r denote the circumradius and inradius of ABC, respectively [42], [24].

Second, as Peter Yff first recognized, Y_1 is the center of one of the four inner Soddy circles. This recognition was based on his having found, in 1978, trilinears for the point and noting, in 1991, that they correspond to those given above [47].

Third, there is a construction found by Noam Elkies: Call two circles within ABC *companion incircles* if they are the incircles of two triangles formed by dividing ABC into two smaller triangles by passing a line through one of the vertices and some point on the opposite side. As proved in [14], any chain of circles $\Gamma_1, \Gamma_2, \ldots$ such that Γ_n, Γ_{n+1} are companion incircles for every n consists of at most six distinct circles; there is a unique chain consisting of only three distinct circles; and for this chain, the three subdividing lines are concurrent in a point, namely the center of an inner Soddy circle. Concerning Soddy circles, see [14, pp. 13–15] and [41].

Y_2 EQUAL DETOUR POINT (CENTER OF AN OUTER SODDY CIRCLE) $\alpha = 1 + 2\sqrt{\mathcal{P}}/a(b+c-a)$, or $\alpha = \sec(A/2)\cos(B/2)\cos(C/2) + 1$ [42]. As Veldkamp explains in his naming of this point in [42], if Y is a point not between A and B, we make a detour of magnitude $|AY| + |YB| - |AB|$ if we walk from A to B via Y; then a point in the plane of ABC is *a point of equal detour* if the magnitudes of the three detours, A to B via Y, B to C via Y, and C to A via Y, are equal. Y_2 is the only such point unless ABC has an angle greater than $2\,\mathrm{Sin}^{-1}(4/5)$, and then Y_1 also has the equal detour property. Yff [47] found that Y_2 is the center of one of the four outer Soddy circles. In accord with (1), Y_2 and Y_1 are harmonic conjugates with respect to X_1 and X_7.

Y_3 **1st MID-ARC POINT** $\alpha = (\cos(B/2) + \cos(C/2))\sec(A/2)$ [26]. Let A', B', C' be the first points of intersection of the angle bisectors of ABC with its incircle Γ. The tangents to Γ at A', B', C' form a triangle $A''B''C''$. The lines AA'', BB'', CC'' concur in Y_3.

Y_4 **2nd MID-ARC POINT** $\alpha = (\cos(B/2) + \cos(C/2))\alpha = (\cos(B/2) + \cos(C/2))\csc A$ [27]. Let A', B', C' be the first points of intersection of the angle bisectors of ABC with its incircle. Let A'', B'', C'' be the midpoints of sides BC, CA, AB, respectively. The lines $A'A'', B'B'', C'C''$ concur in Y_4.

Y_5 **1st MALFATTI POINT** The famous Malfatti problem is to construct three circles $\Gamma_A, \Gamma_B, \Gamma_C$ inside ABC such that each is externally tangent to the other two, Γ_A is tangent to AB and AC, Γ_B is tangent to BC and BA, and Γ_C is tangent to CA and CB. Let A' be the point of contact of Γ_B and Γ_C, B' the point of contact of Γ_C and Γ_A, and C' the point of contact of Γ_A and Γ_B. The lines AA', BB', CC' concur in Y_5. For a discussion of Malfatti's problem, see [17, pp. 244–245], [18, pp. 106–120], and [43, pp. 206–209]. Y_5 originates in [29].

Y_6 **2nd MALFATTI POINT** Let A'', B'', C'' be the excenters of ABC, and let A', B', C' be as in the construction of Y_5. The lines $A'A'', B'B''$, and $C'C''$ meet in Y_6 [29].

Y_7 **CONGRUENT INCIRCLES POINT and other ELKIES POINTS** Y_7 is the point Y for which the triangles BYC, CYA, AYB have congruent incircles. More generally, suppose s_a, s_b, s_c are any three positive numbers. Elkies proved that the interior of ABC contains a unique point Y such that the respective inradii r_a, r_b, r_c of triangles BYC, CYA, AYB satisfy $r_a : r_b : r_c = s_a : s_b : s_c$. His method applies not only to the inradius but to other positive functions of (a, b, c) as well [25].

Y_8 **APOLLONIUS POINT** Let $\Gamma_A, \Gamma_B, \Gamma_C$ be the excircles opposite A, B, C, respectively. Apollonius' (famous) Problem includes the construction of the circle Γ internally tangent to the three excircles and encompassing them. Let A' be the point of contact of Γ and Γ_A, and similarly for B' and C'. The lines AA', BB', CC' concur in Y_8 [1], [17], [28]. See FIGURE 14.

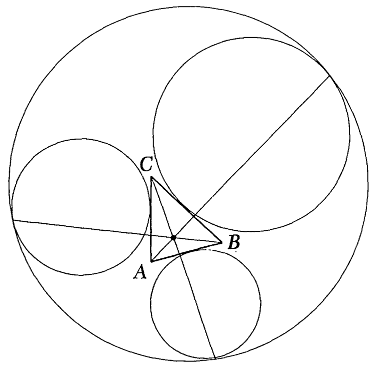

FIGURE 14
The Apollonius point, Y_8, constructed from the circle encircling the excircles.

Y_9 **TRISECTED PERIMETER POINT** $\alpha = bc(v - c + a)(v - a + b)$, **where** v **= unique real root of** $2x^3 - 3(a + b + c)x^2 + (a^2 + b^2 + c^2 + 8bc + 8ca + 8ab)x - (b^2c + c^2a + a^2b + 5bc^2 + 5ca^2 + 5ab^2 + 9abc)$ [47]. There exist points A' on side BC, B' on side CA, and C' on side AB satisfying the perimeter-trisection property, $A'C + CB' = B'A + AC' = C'B + BA'$, such that the lines AA', BB', CC' concur. The point of concurrence is Y_9 (X_8 is the "bisected perimeter point") [4], [7].

Y_{10} **EQUI-BROCARD CENTER** In private correspondence the author conjectured and L. Kuipers proved the existence of a center Y for which the triangles BYC, CYA, AYB have equal Brocard angles.

Y_{12} **1st MORLEY CENTER** $\alpha = \cos A/3 + 2\cos B/3 \cos C/3$ [47]. Morley's Theorem in [10] reads as follows: "The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle." In 1967, Peter Yff determined in unpublished notes that the centroid of Morley's triangle, Y_{12}, is not one of the well-known centers, since its trilinears are not equivalent to "known" trilinears. Morley's triangle appears in FIGURE 17 [37].

Y_{13} **2nd MORLEY CENTER** $\alpha = \sec(A/3)$ [47]. In connection with his work involving Y_{12}, Yff proved that Morley's triangle is perspective with the reference triangle ABC. Y_{13} is the center of perspective.

5. Constant Distance-ratios

It is well known that the centroid lies 2/3 of the way from the orthocenter to the circumcenter; that is, $X_4X_2 : X_4X_3 = 2:3$, where XY denotes the directed distance from X to Y. It is also known that the nine-point center and the De Longchamps point participate in the chain of constant distance-ratios indicated in FIGURE 15.

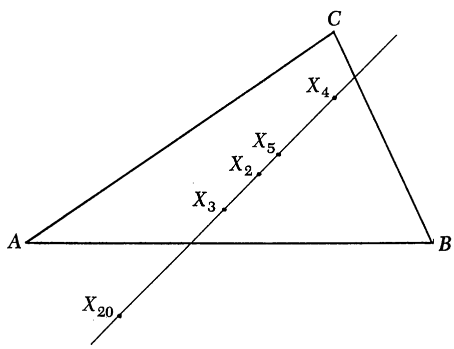

FIGURE 15
The Euler line, along which X_4X_5: X_4X_2: X_4X_3: X_4X_{20} = 3: 4: 6: 12.

Among the hundreds of central lines formed by pairs of the 101 centers in Table 1, there are only 16 lines along which constant distance-ratios occur. These are shown in Table 3.

TABLE 3

X_1X_2: X_1X_{10}: $X_1X_8 = 2:3:6$	X_1X_3: $X_1X_{40} = 1:2$
X_1X_{46}: $X_1X_{56} = 2:1$ (X_{46} = reflection of X_1 about X_{56})	X_1X_{11}: $X_1X_{80} = 1:2$
X_2X_6: $X_2X_{69} = 1: -2$ (X_2 separates X_6 and X_{69})	X_2X_7: $X_2X_9 = 2: -1$
X_2X_{11}: $X_2X_{100} = 1: -1$	X_2X_{37}: $X_2X_{75} = 1: -2$
X_2X_{39}: $X_2X_{76} = 1: -2$	X_3X_{98}: $X_3X_{99} = 1: -1$
X_4X_5: X_4X_2: X_4X_3: $X_4X_{20} = 3:4:6:12$	X_5X_{51}: $X_5X_{52} = 1:3$
X_4X_{10}: $X_4X_{40} = 1:2$	X_8X_{20}: $X_8X_{40} = 2:1$
$X_{10}X_{65}$: $X_{10}X_{72} = 1: -1$	$X_{10}X_{80}$: $X_{10}X_{100} = 1: -1$

The entries in Table 3 that involve centers X_1, X_3, X_{40} and X_1, X_{46}, X_{56} deserve special mention: These five centers are collinear; yet, the *only* constant distance-ratios involving these centers are those indicated in Table 3 [e.g., the ratio $X_1X_3 : X_1X_{46}$ *varies* as (a, b, c) varies].

6. Topics for Undergraduate Research

Successful undergraduate research depends on subject matter in which plenty that is new and interesting *will* be found by inquisitive, able, persistent undergraduate students. Here, we mention several lines of inquiry that should prove fertile for undergraduate research assisted by computing.

General triangles A perusal of Table 1 shows that there are several named triangles derived from ABC and some particular center. For example, the excentral triangle is to X_1 as the anticomplementary is to X_2 and the tangential is to X_6; the medial is to X_2 as the orthic is to X_4. These observations point toward a unification of some basic ideas of triangle geometry, even though we must step outside geometry and into algebra to carry out this unification. Define the *cevian triangle* of a center $X = \alpha : \beta : \gamma$ as the triangle whose vertices are $0 : \beta : \gamma, \alpha : 0 : \gamma, \alpha : \beta : 0$ and the *anticevian triangle* of X as the triangle whose vertices are $-\alpha : \beta : \gamma, \alpha : -\beta : \gamma, \alpha : \beta : -\gamma$.

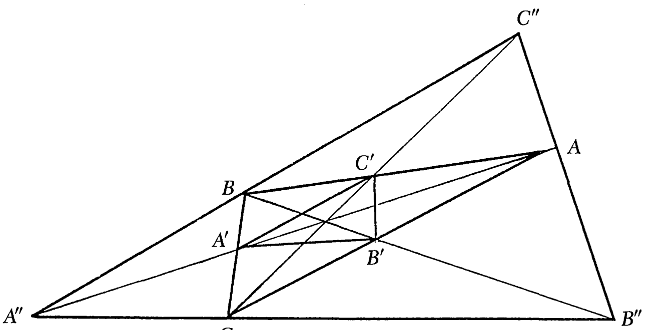

FIGURE 16
$A'B'C'$ is the cevian triangle of X; $A''B''C''$ is the anticevian triangle of X. By (1), X and A'' are harmonic conjugates with respect to A and A', and similarly along lines BX and CX.

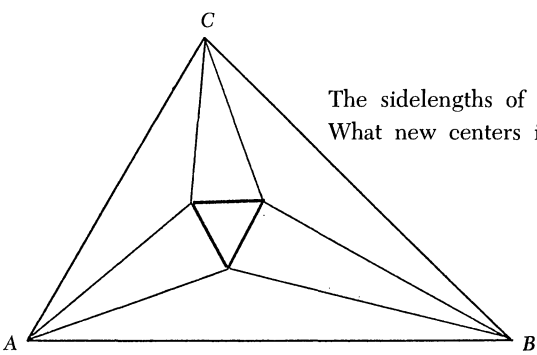

FIGURE 17
The sidelengths of Morley's Triangle are $8R \sin\frac{A}{3} \sin\frac{B}{3} \sin\frac{C}{3}$. What new centers involving $\frac{A}{3}, \frac{B}{3}, \frac{C}{3}$ await discovery?

For example, the medial and orthic triangles are cevian triangles, and the anticomplementary, tangential, and excentral are anticevian triangles. Students can write appropriate computer programs that compute trilinears of centers of derived triangles. While that is good exercise, it certainly is not an end in itself. Instead, it provides fertile grounds on which students can formalize their own questions and pursue answers. Which centers so obtained are interesting for one reason or another (e.g., $X_9, X_{10}, X_{20}, X_{26}, X_{51}, X_{52}, X_{53}$)? Students can also seek pairs, or sets, of perspective triangles. Which centers of perspective are interesting (e.g., X_{19}, X_{24}, X_{25})? For what pairs of perspective triangles must the axis of perspective be a central line? Similar questions accompany other families of triangles, such as the pedal and antipedal triangles formulated in Section 2.

Curves Among "central curves" in the literature are the power curve a^t: b^t: c^t ([**20**], [**23**]) and Kiepert's hyperbola $\csc(A+t)$: $\csc(B+t)$: $\csc(C+t)$, where t varies through the real numbers, as well as special circles and conic sections. Undergraduates can create, explore, and graph such curves on computers. Some will be inclined to extend their work to curves and loci of their own devising. (Kiepert's hyperbola occurs in Gallatly [**19**, pp. 139 & 151] and Casey [**6**, pp. 442–449].) [Ed. Note: See this issue, p. 188.]

Betweenness Table 3 shows several cases in which a particular center always lies on the line of two other particular centers and always *between* them. In those cases, however, the distance-ratios between pairs of such centers remain constant over all (a, b, c) in $\mathbb{T}$. Using original computer programs, cases of betweenness where the distance-ratios need not stay fixed. There also exist orderings of centers according to their distances *away from* a particular line. There are cases in which two particular centers always (or never) lie on the same side of a particular central line.

Mappings Isogonal conjugation $(\alpha{:}\beta{:}\gamma \to \alpha^{-1}{:}\beta^{-1}{:}\gamma^{-1})$ maps the interior of ABC onto itself, and correspondingly transforms the rest of the plane excluding the sidelines BC, CA, AB. It is well known that this mapping carries lines onto conic sections that circumscribe ABC (e.g., [**16**]). Students can find and explore other center-preserving mappings.

Functional equations There exist many problems of the form "Find *all* centers (or *all regular* centers, etc.) that have a certain geometric (or algebraic) property." Here is an example. For any center $\boldsymbol{X}$, let X, X_A, X_B, X_C denote the values of $\boldsymbol{X}$ at the triangles ABC, XBC, AXC, ABX, respectively; find all $\boldsymbol{X}$ for which the triangle $X_AX_BX_C$ is perspective with ABC. The solution set includes X_1, X_2, X_3, X_4, X_5, $X_6, X_{20}, X_{31}, X_{32}, X_{75}, X_{76}$, but no other centers listed in Table 1.

Centrifugal centers The word *centrifugal* here applies to any center X which moves to infinity as ABC becomes equilateral. That is, as $a:b:c \to 1:1:1$, the center X moves away from X_1 so that the distance $|XX_1|$ increases without bound. Several centrifugal centers appear in Table 1. Students may seek others and general conditions on $f(a, b, c)$ that force the center $f(a, b, c)$: $f(b, c, a)$: $f(c, a, b)$ to be centrifugal. A related problem is to determine all possible limiting behaviors of centers as $a:b:c$ tends to $1:1:1$.

Elkies points Suppose Y is a center whose trilinears are always positive, so that Y lies inside ABC. Let α': β': γ' be any triple proportional to the respective inradii of the triangles BYC, CYA, AYB, as described at Y_7 in Table 2. Call these *homogeneous triradial coordinates* (*triradials*) for Y. Is there a reasonable formula or algorithm for transforming trilinears to triradials, or the reverse? Can criteria for collinearity, concurrence of lines, and so on, be found in terms of triradials? (Note that if "inradii" is replaced by "areas" the resulting coordinate system is barycentric.)

Tetrahedra Analogous to trilinears relative to a triangle are *quadriplanar coordinates* relative to a tetrahedron. Centers, central lines, and central planes await computer-assisted discovery and development. For quadriplanar coordinates, see Woods [**44**, pp. 193–196]; for much else, see Altshiller-Court [**2**]; for inequalities and further references, see [**36**, Chapter XIX].

Morley's Triangle In connection with the Morley points, Y_{12} and Y_{13}, one wonders what other new centers can be derived from Morley's Triangle. Note that of all the α's in Section 3, the one-third angles, $A/3, B/3, C/3$, occur only in connection with Y_{12} and Y_{13}. An excellent introduction to Morley's Triangle(s), including 150 references, is [**37**].

Minima and Maxima As mentioned in Section 3, X_6 and X_{13} minimize certain simple sums involving distances. Another such case, not found in the literature but straightforwardly verifiable, is that X_9 (α: β: $\gamma = b+c-a$: $c+a-b$: $a+b-c$) maximizes the function $\beta\gamma + \gamma\alpha + \alpha\beta$. What other centers minimize or maximize symmetric functions of α, β, γ?

Experimental mathematics The findings reported in this article and the suggested topics for undergraduate research typify the sort of experimental mathematics that is made possible by computing. A lively thought along these lines was expressed by Klaus Peters in an announcement of the coming of *The Journal of Experimental Mathematics*, "What we are talking about is nothing less than a possible revolution in the way mathematicians think about and report their work. Anyone familiar with the standards of experimentation in other branches of science can only wonder what standards will evolve for judging experimental work in mathematics."

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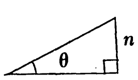
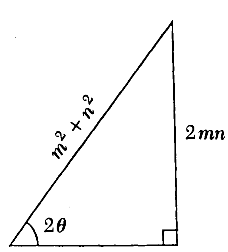
Proof without Words: Pythagorean Triples via Double Angle Formulas



$m > n > 0$
$m, n \in \mathbf{I}$

$$\begin{cases} \sin\theta = \dfrac{n}{\sqrt{m^2+n^2}} \\ \cos\theta = \dfrac{m}{\sqrt{m^2+n^2}} \end{cases}$$

$$\left. \begin{aligned} \sin 2\theta &= \frac{2mn}{m^2+n^2} \\ \cos 2\theta &= \frac{m^2-n^2}{m^2+n^2} \end{aligned} \right\}$$



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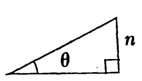

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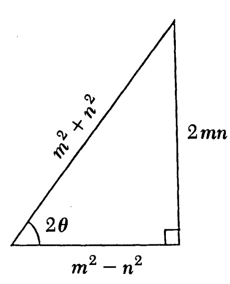

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The Conics of Ludwig Kiepert: A Comprehensive Lesson in the Geometry of the Triangle

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1. Introduction

If a visitor from Mars desired to learn the geometry of the triangle but could stay in the earth's relatively dense atmosphere only long enough for a single lesson, earthling mathematicians would, no doubt, be hard-pressed to meet this request. In this paper, we believe that we have an optimum solution to the problem. The Kiepert conics, though seemingly unknown today, constitute a significant part of the geometry of the triangle and to study them one has to deal with many fundamental concepts related to this geometry such as the Euler line, Brocard axis, circumcircle, Brocard angle, and the Lemoine line in addition to well-known points including the centroid, circumcentre, orthocentre, and the isogonic centres. In the process, one comes into contact with not so well known, but no less important concepts, such as the Steiner point, the isodynamic points and the Spieker circle.

In this paper, we show how the Kiepert's conics are derived using both analytic and projective arguments and discuss their main properties, which we have drawn together from several sources. We have applied some modern technology, in this case computer graphics, to produce a series of pictures that should serve to increase the reader's appreciation for this interesting pair of conics. In addition, we have derived some results that we were unable to locate in the available literature.

2. Preliminaries

a. Coordinate Systems Two systems of specialized homogeneous coordinates are especially suited to this type of work, they are, the "trilinear" (or normal) system and the "barycentric" (or areal) system. In the trilinear case, the coordinates (x, y, z) of a point P in the plane of a given reference triangle ABC are proportional to the signed distances of P from the sides of the reference triangle, i.e.

$$x : y : z = d_a : d_b : d_c,$$

where, for example, d_a represents the *signed* distance of P from side BC. The sign of d_a is positive or negative accordingly as P and the unit point, the incentre $I = (r, r, r) = (1, 1, 1)$, r is the inradius, are on the same or opposite sides of BC. The actual distances d_a, d_b, d_c of a point P from the sides of ABC are related to the trilinear coordinates of P by the equations:

$$\frac{d_a}{x} = \frac{d_b}{y} = \frac{d_c}{z} = \frac{r(a+b+c)}{ax+by+cz},$$

see Sommerville [**32**, p. 157].

For the trilinear line coordinates $[u, v, w]$ of the line l: $ux + vy + wz = 0$, one has

$$u : v : w = ad_A : bd_B : cd_C,$$

where d_A represents the signed distance from vertex A to l. The signs of d_A and d_B, for example, are the same or different depending on whether or not A is in the half-plane determined by B and l.

For the barycentric type, the coordinates of P are proportional to the signed areas of the triangles PBC, PCA, PAB thus,

$$x : y : z = ad_a : bd_b : cd_c,$$

where the unit point of the system is now the centroid G.

Similarly, for the line case, one has

$$u : v : w = d_A : d_B : d_C.$$

Unless otherwise indicated, we shall use the trilinear system throughout. A curious application of these coordinates, the three jug-problem, is given by Coxeter and Greitzer [**11**, pp. 89–93].

b. Transformations We shall have occasion to use two special cases of so-called "Cremona" transformations, named after the Italian geometer Luigi Cremona (Pavia, 1830–Rome, 1903) who did considerable study on them, see Coolidge [**7**, pp. 287ff]. The Cremona transformations are birational transformations of the plane. The following quadratic type given by the systems of equations

$$(x', y', z') = \left(\frac{n_1}{x}, \frac{n_2}{y}, \frac{n_3}{z}\right) = (n_1 yz, n_2 zx, n_3 xy), \tag{1}$$

where n_i, $i = 1, 2, 3$, are nonzero constants, is the only one of interest at this time.

These transformations induce involutions on the points in the plane not on the sides of the reference triangle and carry, in a one-to-one fashion, lines into conics that pass through the vertices of the reference triangle and vice versa. As mentioned earlier, we are interested in two special cases of (1).

Case 1. The quadratic transformation

$$P = (x, y, z) \mapsto P' = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \tag{2}$$

is obtained from (1) by making the substitutions $n_1 = n_2 = n_3 = 1$. Geometrically, P' is obtained from P by reflecting the lines AP, BP, CP in the internal angle bisectors through A, B, C respectively: The reflected lines concur in the point P'. This point which we shall now denote by P^g, is called the *isogonal conjugate* of P and the transformation g defined by (2), the *isogonal* transformation. See Kimberling [**18**] for some familiar pairs of points related by this transformation.

A well-known pair of this sort is (K, G) where K denotes the symmedian point and G the centroid. The *symmedian point* of a triangle ABC is the point of concurrence of the *symmedians*, i.e., the reflections of the medians with respect to the angle bisectors. Thus, it is more or less by definition the isogonal conjugate of the centroid. But the symmedian point can also be defined as the point inside the triangle such that the sum $d_a^2 + d_b^2 + d_c^2$ is minimum. In fact, according to Mackay [**21**], the first appearance of this point in the mathematical literature was in the context of this very property. The symmedian point is usually referred to as the *Lemoine point* by French and British writers but is known as the *Grebe point* in Germany, see Johnson [**14**, p. 213]. Émile Michel Hyacinthe Lemoine (Quimper, 1840–Paris, 1912), one of the main promotors of the modern geometry of the triangle, had first been a teacher of mathematics and then, from 1870, Engineering Advisor at the Court of Commerce of Paris; Ernst Wilhelm Grebe (Michelbach near Marburg, 1804–Kassel, 1874) was a teacher of mathematics at the Gymnasium in Kassel. A considerable amount of research relating to the symmedian point seems to have been carried out during the 19th century.

Case 2. The quadratic transformation

$$P = (x, y, z) \mapsto P' = \left(\frac{1}{a^2 x}, \frac{1}{b^2 y}, \frac{1}{c^2 z} \right), \tag{3}$$

is obtained from (1) by making the substitutions $n_1 = 1/a^2, n_2 = 1/b^2, n_3 = 1/c^2$. Geometrically, P' is obtained from P by reflecting the points D, E, F (the intersections of the lines AP, BP, CP with the sides BC, CA, AB) in the midpoints of the side on which they lie: The lines AD', BE', CF' where D', E', F' denote the respective images of D, E, F concur in the point P'. This point, which we shall now denote by P^t, is called the *isotomic conjugate* of P and the transformation t defined by (3), the *isotomic* transformation.

A familiar pair of points related by this transformation is formed by the Gergonne point and the Nagel point. The *Gergonne point* of a triangle is the point of concurrence of the line segments connecting the vertices with the points of contact of the incircle with the opposite sides. The definition of the *Nagel point* is similar except now we consider the points of contact of the three excircles. A discussion of both of these points is given in Johnson [**14**, pp. 184–185]; their trilinear coordinates can be found in Kimberling [**18**]. Joseph Diaz Gergonne (Nancy, 1771–Montpellier, 1859) founded, in 1810, the first only wholly mathematical journal, the *Annales de mathématiques pures et appliquées*; he held the chair of astronomy at the University of Montpellier, where he also acted as rector. Christian Heinrich (von) Nagel (Stuttgart, 1803–Ulm, 1882, nobled in 1875) was Professor for Mathematics and Science at the Gymnasium in Ulm and director of a so-called "Realschule". He devoted a lot of his activities to a modernization of the school system.

3. The Hyperbola

The following problem was proposed in 1868 by Lemoine [**19**].

> *Construire un triangle, connaissant les sommets des triangles équilatéraux construits sur les côtes.* (Construct a triangle, given the peaks of the equilateral triangles constructed on the sides.)

A solution by Friedrich Wilhelm August Ludwig Kiepert (Breslau, 1846–Hannover, 1934) was published in 1869 [**17**]. At the time, Kiepert was a doctoral student at the University of Berlin under Weierstraß. He later moved to Hannover as Professor of Higher Mathematics and became Dean in 1901. He wrote a textbook on calculus that had been used frequently in German universities up to the 1920s. His later mathematical work was concerned mainly with actuarial theory. For more information on this mathematician, see Volk [37].

Kiepert's solution contains a remark that we shall cast in the form of a theorem. In order to avoid degenerate cases, we fix a triangle ABC that is assumed, for the remainder of this paper, to be scalene with $\alpha > \beta > \gamma$, where α, β, γ denote the measures of the angles at the vertices A, B, C respectively. In addition, this will be the reference triangle when computations with coordinates are carried out.

THEOREM 1. *If the three triangles $A'BC$, $AB'C$ and ABC', constructed on the sides of the given triangle ABC as bases, are similar isosceles and similarly situated, then the lines AA', BB', CC' concur at a point P. The locus of P as the base angle varies between $-\pi/2$ and $\pi/2$ is the conic*

$$\boldsymbol{\Gamma}\colon \frac{\sin(\beta-\gamma)}{x} + \frac{\sin(\gamma-\alpha)}{y} + \frac{\sin(\alpha-\beta)}{z} = 0, \tag{4}$$

or, equivalently,

$$\boldsymbol{\Gamma}\colon \frac{bc(b^2-c^2)}{x} + \frac{ca(c^2-a^2)}{y} + \frac{ab(a^2-b^2)}{z} = 0. \tag{5}$$

Proof. We denote the measure of the base angles of the similar, isosceles triangles by ϕ (noting that the orientation of these triangles is counterclockwise when $\phi < 0$ and clockwise when $\phi > 0$) and immediately obtain the following representations for the vertices of the triangle $A'B'C'$,

$$\begin{aligned} A' &= (-\sin\phi, \sin(\gamma+\phi), \sin(\beta+\phi)), \\ B' &= (\sin(\gamma+\phi), -\sin\phi, \sin(\alpha+\phi)), \\ C' &= (\sin(\beta+\phi), \sin(\alpha+\phi), -\sin\phi). \end{aligned}$$

Now, recall that the trilinear (or barycentric) coordinates $[u, v, w]$ of the line $ux + vy + wz = 0$ connecting the two given points can be taken as the cross product of the two vectors in $\mathbb{R}^3$ whose components are the trilinear (barycentric) coordinates of these points. Since $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, we readily derive the representations

$$\begin{aligned} AA' &= [0, -\sin(\beta+\phi), \sin(\gamma+\phi)], \\ BB' &= [\sin(\alpha+\phi), 0, -\sin(\gamma+\phi)], \\ CC' &= [-\sin(\alpha+\phi), \sin(\beta+\phi), 0]. \end{aligned}$$

The point P, shown in FIGURE 1, is easily seen to have the coordinates

$$\begin{aligned} (x,y,z) = (\sin(\beta+\phi)\sin(\gamma+\phi), &\sin(\gamma+\phi)\sin(\alpha+\phi), \\ &\sin(\alpha+\phi)\sin(\beta+\phi)), \end{aligned} \tag{6}$$

which implies

$$x\sin(\alpha+\phi) = y\sin(\beta+\phi) = z\sin(\gamma+\phi).$$

It now follows that

$$(x\sin\alpha - y\sin\beta)\cos\phi + (x\cos\alpha - y\cos\beta)\sin\phi = 0$$

and

$$(x\sin\alpha - z\sin\gamma)\cos\phi + (x\cos\alpha - z\cos\gamma)\sin\phi = 0,$$

a system of homogeneous equations linear in the variables $\cos\phi, \sin\phi$. Since each triple (x, y, z) gives a nontrivial solution, the determinant must vanish, hence

$$(x\sin\alpha - y\sin\beta)(x\cos\alpha - z\cos\gamma) = (x\sin\alpha - z\sin\gamma)(x\cos\alpha - y\cos\beta),$$

which completes the proof.

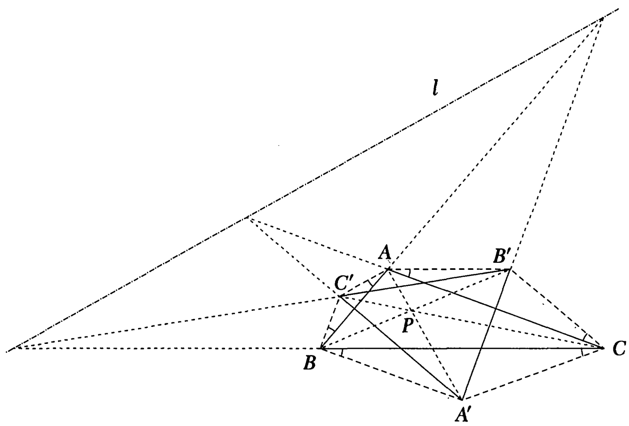

FIGURE 1

In order to determine the conic type, we inspect the line at infinity. If we consider the trilinear line coordinates as described in Section 2a, it is clear that, for a line far distant from the reference triangle, the distances d_A, d_B, d_C become almost equal, consequently, in the limit, we obtain $[a, b, c]$ as the line coordinates of the line at infinity. Furthermore, an examination of the intersection of this line with Γ reveals that the relevant discriminant reduces to the expression $E = (a^2 + b^2 + c^2)^2 - 3(a^2b^2 + b^2c^2 + c^2a^2)$. Since $E > 0$, see Bottema et al. [2, p. 11], the conic under consideration meets the line at infinity in two distinct points; thus, it is a hyperbola, *Kiepert's hyperbola*. The significance of the line l in FIGURE 1 will be explained later, in Theorem 2.

To gain more insight into the angles that actually produce the points at infinity, we note that these points correspond to angles ϕ satisfying the equation

$$\frac{\sin\alpha}{\sin(\alpha+\phi)} + \frac{\sin\beta}{(\sin\beta+\phi)} + \frac{\sin\gamma}{\sin(\gamma+\phi)} = 0.$$

By means of the usual formulae and theorems ($\sin\alpha = a/2R, \ldots, R = abc/4\Delta$, R the circumradius, Δ the area of the reference triangle, law of cosines, Heron's formula for the area of a triangle) this equation can be transformed into

$$\sin(2\phi + \omega) = -2\sin\omega,$$

where the angle ω is the Brocard angle, determined by the property that there is a (unique) point τ_1, *Brocard's first point*, such that $\measuredangle \tau_1 CA = \measuredangle \tau_1 AB = \measuredangle \tau_1 BC = \omega$, see Kimberling [18]. Pierre René Jean-Baptiste Henri Brocard (Vignot 1845–1922 Barle-Duc) is, like Lemoine, regarded one of the fathers of the modern geometry of the triangle. He was not a professional mathematician but, rather, served as an army officer for engineering in Algier and Montpellier. His widespread results form the basis for what is now known as *Brocard geometry*.

Since $0 < \sin\omega = 2\Delta/\sqrt{a^2b^2 + b^2c^2 + c^2a^2} < 1/2$ for a scalene triangle, this equation has two solutions in the range required, namely

$$\max(-\pi/2, -\alpha) < \phi_1 < -\beta < -\phi_2 < -\gamma.$$

The following table indicates some special points on the hyperbola corresponding to certain specific values of ϕ:

Measures of ϕ	Points
0	centroid (G)
$\frac{\pi}{2}$	orthocentre (H)
$-\alpha$, if $\alpha \leq 90°$; $180° - \alpha$, if $\alpha > 90°$	A
$-\beta$	B
$-\gamma$	C
$\frac{\pi}{3}$	Fermat point (F_1) (first isogonic centre)
$-\frac{\pi}{3}$	second isogonic centre (F_2)
ω	τ^g
$-\omega$	Brocard's third point (τ_3)

Since some of the cases above may not be familiar, we include the following explanatory remarks:

(i) The *Fermat point* is the point inside the triangle (provided no angle exceeds 120°) such that the sum $\overline{AP} + \overline{BP} + \overline{CP}$, $P \in ABC$, is minimum, see Johnson [**14**, p. 221]. Nicholas D. Kazarinoff presents an interesting alternative treatment of this point using an elementary idea of statics [**15**, pp. 117–118].

(ii) In addition to Brocard's first point, there is *Brocard's second point*, the unique point τ_2 such that $\sphericalangle AB\tau_2 = \sphericalangle BC\tau_2 = \sphericalangle CA\tau_2 = \omega$, see Lemoine [**20**], and Kimberling [**18**]. The point τ in the table is the midpoint of the segment $\tau_1\tau_2$, called *Brocard midpoint* by Kimberling [**18**]. An added significance for τ will be given later when various properties of the hyperbola are discussed.

(iii) The point τ_3 in the table, *Brocard's third point* will be discussed more explicitly in Section 5. The triangle $A'B'C'$ corresponding to $\phi = -\omega$ is *Brocard's first triangle*, see [**14**, pp. 277–280]. We refer to this triangle again when the parabola is discussed. Other points important in Brocard geometry occur on the hyperbola by taking other measures of ϕ that involve ω. We shall not dwell on these here, instead, we refer the interested reader to the paper of M'Cay [**23**].

At this point we make a reference to a property relating to the case when $\phi = \pi/3$ that is generally attributed to Napoleon Bonaparte (1769–1821) [**9**, p. 23], which reads

> *The circumcircles of the triangles $A'BC$, $AB'C$ and ABC' meet at the Fermat point F_1 and their centres form a fourth equilateral triangle.*

In fact, this would appear to be the starting point for the whole story. Equilateral triangles being erected on the faces of an arbitrary triangle appeared first in the context of Napoleon's Theorem. For more information, see the recent papers by Schmidt [**30**] and Wetzel [**38**], where the theorem is traced up to 1825. One can thus assume that it was known to Lemoine and, most likely, served as the basis for his initial question, which was answered by Kiepert.

We now describe the course of the hyperbola more precisely. Assume the triangle ABC to be acute-angled with $\alpha > \beta > (\pi/3)\gamma > \omega$. Let us start out with $\phi = -\pi/2$, in which case the point P that traces out the hyperbola is at the orthocentre. The first notable value of ϕ is $\phi = -\alpha$ whereby $P = A$. Then P moves to infinity, passes through B when $\phi = -\beta$, through F_2 when $\phi = -\pi/3$ and back again to infinity. The remaining values of interest are $\phi = -\gamma$ $(P = C)$, $\phi = -\omega$ $(P = \tau_3)$, $\phi = 0$ $(P = G)$, $\phi = \omega$ $(P = \tau^g)$, $\phi = \pi/3$ $(P = F_1)$ and finally $\phi = \pi/2$ when P returns to the orthocentre, see FIGURE 2. The course changes somewhat when ABC is obtuse-angled, see FIGURE 3.

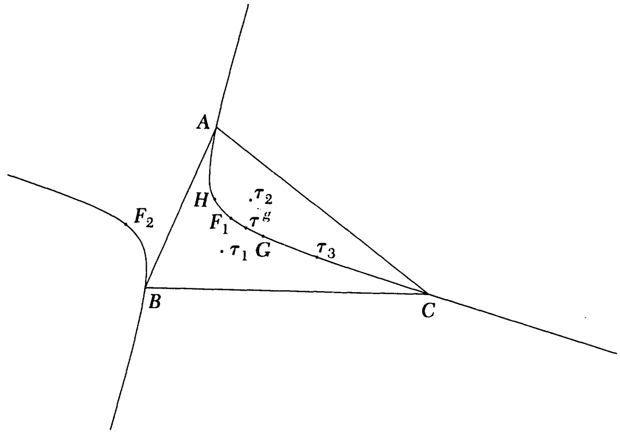

FIGURE 2

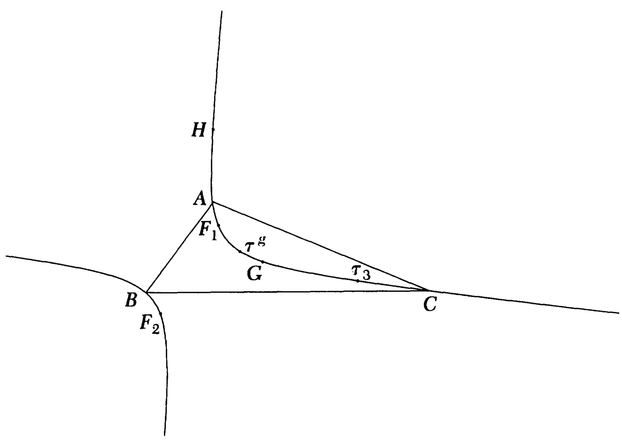

FIGURE 3

A projective derivation of Γ is possible by considering the points A' and B' as elements of the point ranges represented by the perpendicular bisectors of BC and CA respectively. Since the triangles $A'BC$ and $AB'C$ are similar, any four positions of A' have the same cross ratio as the four corresponding position of B'. The point $P = AA' \cap BB'$ is thus the intersection of corresponding elements of two projectively related pencils centred at A and B and hence its locus is a conic through A and B [36, pp. 109ff]. This derivation was considered by, among others, Frederick G. Maskell and Jordi Dou [22].

Kiepert's hyperbola has a number of interesting properties that serve to emphasize its importance in the geometry of the triangle. We now summarize those that would seem to be most accessible to the general reader.

(i) It is rectangular (asymptotes are perpendicular) and its centre lies on the nine-point circle. This is an immediate consequence of the following theorem that, although attributed to Karl Wilhelm Feuerbach (1800–1834) by Cooledge [6, p. 123], can not be found in Feuerbach's book of 1822 [13]. The claim on the centre had been proved earlier in 1821, by Charles-Julien Brianchon (1783–1864) and Jean-Victor Poncelet (1788–1867) [4, THEOREM VII].

The locus of the centres of all conics through the vertices and orthocentre of a triangle, which conics, when not degenerate, are rectangular hyperbolas, is a circle through the middle points of the sides, the points half-way from the orthocentre to the vertices, and the feet of the altitudes.

Moreover, the centre of the hyperbola is midway between the isogonic centres of ABC, see FIGURE 4.

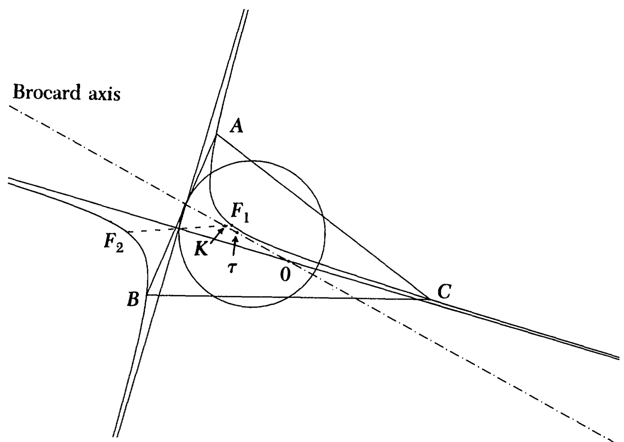

FIGURE 4

(ii) The image Γ^g of Kiepert's hyperbola, under the isogonal transformation, has the equation

$$\Gamma^g\colon \sin(\beta-\gamma)x+\sin(\gamma-\alpha)y+\sin(\alpha-\beta)z=0, \tag{7}$$

or, equivalently,

$$\Gamma^g\colon bc(b^2-c^2)x+ca(c^2-a^2)y+ab(a^2-b^2)z=0, \tag{8}$$

which represents the *Brocard axis* of ABC defined as the line connecting the symmedian point to the circumcentre. This line is perpendicular to the *Lemoine line*, i.e., the axis of the perspectivity of the triangle ABC and its *tangential triangle* $A_tB_tC_t$ formed by the tangents to the circumcircle at the vertices. The name "Lemoine line" is justified by the fact that the Lemoine point—see Section 2b, Case 1—is the centre of this perspectivity. The Brocard axis contains, in addition to the symmedian point and the circumcentre, the isodynamic points F_1^g, F_2^g (which also are the common points of the circles of Apollonius), the Brocard midpoint τ discussed previously, and at least seven more noteworthy points of the reference triangle, see Kimberling [18].

(iii) One may also make use of the isogonal transformation when looking for the asymptotes. To this end, recall first the notion of the Wallace-Simson line of a point on the circumcircle. The feet of the perpendicular lines from a point P to the sides of a triangle are collinear if, and only if, P belongs to the circumcircle of the triangle in which case the resulting line is called the *Wallace-Simson line* of P, see [11]. The geometric description given for the isogonal transformation shows that the isogonal transform of the circumcircle is the line at infinity and the Wallace-Simson line of any point on the circumcircle of the reference triangle passes through the isogonal conjugate of its diametral point. Furthermore, a tedious but straightforward, computation shows that the asymptotes of any circumscribed equilateral hyperbola are the Wallace-Simson lines of the isogonal conjugates of its points at infinity, i.e., the intersection points of the isogonal transform of the hyperbola with the circumcircle. From this we conclude that the Wallace-Simson lines of the intersections P, Q of the Brocard axis with the circumcircle of ABC are the asymptotes of Kiepert's hyperbola, see FIGURE 4. Further treatments of the asymptotes and the centre are given by Mineur [25] and Rigby [29].

(iv) Since the coordinates of the nine-point centre are

$$(\cos(\beta-\gamma), \cos(\gamma-\alpha), \cos(\alpha-\beta)),$$

see [32, p. 159], it follows from the equation

$$\begin{pmatrix} 0 & \sin(\alpha-\beta) & \sin(\gamma-\alpha) \\ \sin(\alpha-\beta) & 0 & \sin(\beta-\gamma) \\ \sin(\gamma-\alpha) & \sin(\beta-\gamma) & 0 \end{pmatrix} \begin{pmatrix} \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) \end{pmatrix} = \begin{pmatrix} \sin(\gamma-\beta) \\ \sin(\alpha-\gamma) \\ \sin(\beta-\alpha) \end{pmatrix}$$

that the polar of the nine-point centre with respect to Kiepert's hyperbola is the Brocard axis. For the basic properties of poles and polars, see Somerville [32, pp. 26–28].

(v) Another connection of Kiepert's hyperbola with the nine-point circle is found in [5, p. 459]. If one considers the triangle formed by the tangents to Γ at the vertices A, B, C, then the orthocentre of this triangle is the centre of the nine-point circle. Casey attributes this property to Brocard.

(vi) A more recent "rediscovery" of Kiepert's hyperbola is given in the following problem of Bottema and van Hoorn [3].

> *Let P be a point in the plane of a nonequilateral triangle ABC and let π be the trilinear polar (or harmonical) of P with respect to ABC. Show that the locus of the points P, such that π is perpendicular to the Euler line of ABC, is a rectangular hyperbola passing through the vertices of ABC, through its centroid and through its orthocentre.*

Here, the *trilinear polar* of a point P is the axis of perspectivity of the triangles ABC and DEF (D, E, F are again the intersections of the lines AP, BP, CP with the sides BC, CA, AB). If $P=(p,q,r)$, the trilinear polar is the line with coordinates $[qr, rp, pq]$, see [10, p. 185]. While this is an interesting and somewhat different aspect of Kiepert's hyperbola, the result is not new. A reference to this particular property may be found in M'Cay [23].

(vii) Another recent rediscovery of the hyperbola is given by Courcouf [8] as an application of areal coordinates to the geometry of the triangle. Here as in the case of the previous problem, see [24], the name Kiepert is not associated with this conic. This is further evidence that the Kiepert conics are not well known today.

(viii) A subtle comment on the centre of Γ is given by Thébault [34]. Let angular bisectors of the reference triangle meet BC, CA, AB in the points A_1, B_1, C_1. Let $A_1^{\#}$ be the harmonic conjugate of A_1 with respect to B and C, and let $B_1^{\#}$ and $C_1^{\#}$ be defined in a similar manner. Thébault's result is that the circles $A_1B_1C_1$, $A_1B_1^{\#}C_1^{\#}$, $B_1C_1^{\#}A_1^{\#}$, $C_1A_1^{\#}B_1^{\#}$ meet at the centre of Γ.

(ix) A novel treatment of the hyperbola using a complex-number approach is given by Kelly and Merriell [16]. In the notation of our Theorem 1, the authors show that the perpendiculars from A to $B'C'$, B to $C'A'$, C to $A'B'$ concur and the locus of the point of concurrence with the trilinear coordinates $(1/\cos(\alpha-\phi), 1/\cos(\beta-\phi), 1/\cos(\gamma-\phi))$, as ϕ varies, is Kiepert's hyperbola, see FIGURE 5.

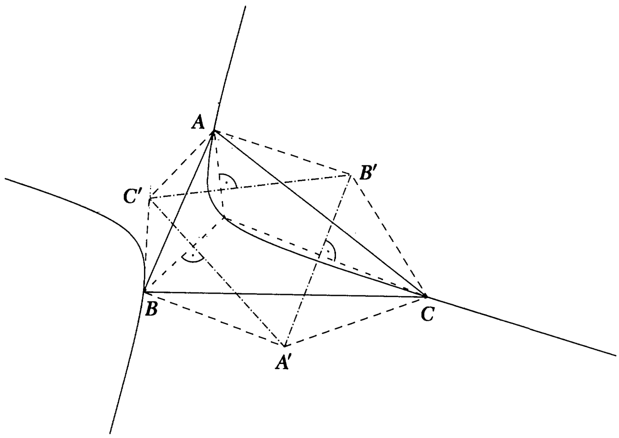

FIGURE 5

(x) Vanderghen [35] notes that Kiepert's hyperbola is the cevian transform (isotomic conjugate) of the tangent to Γ at the centroid G. To see this consider the alternative form of Γ given by (5). Since the coordinates of G are (bc, ca, ab), the line coordinates of the tangent at this point are given by

$$\begin{pmatrix} 0 & ab(a^2-b^2) & ca(c^2-a^2) \\ ab(a^2-b^2) & 0 & bc(b^2-c^2) \\ ca(c^2-a^2) & bc(b^2-c^2) & 0 \end{pmatrix}\begin{pmatrix} bc \\ ca \\ ab \end{pmatrix} = \begin{pmatrix} a(c^2-b^2) \\ b(a^2-c^2) \\ c(b^2-a^2) \end{pmatrix}; \quad (9)$$

these are the line coordinates of the isotomic conjugate of Γ.

(xii) If a *point conic* F is given by the equation $\sum_{i,j} a_{ij} x_i x_j = 0$; $i, j = 1, 2, 3$, then it is an easy exercise to verify that the corresponding *line* form f of F is defined by the equation $\sum_{i,j} A_{ij} u_i u_j = 0$, where (A_{ij}) is the adjoint of (a_{ij}). Thus for the hyperbola, the line form is

$$\gamma\colon p^2u^2 + q^2v^2 + r^2w^2 - 2pquv - 2qrvw - 2rpwu = 0,$$

where $(p, q, r) = (bc(b^2-c^2), ca(c^2-a^2), ab(a^2-b^2))$ and $[u, v, w]$ is a tangent to Γ.

4. The Parabola

In order to introduce the second conic, we state and prove the following:

THEOREM 2. *The envelope of the axis of the triangles ABC and A'B'C' is the parabola*

$$\Sigma\colon \frac{\sin\alpha(\sin^2\beta - \sin^2\gamma)}{u} + \frac{\sin\beta(\sin^2\gamma - \sin^2\alpha)}{v} + \frac{\sin\gamma(\sin^2\alpha - \sin^2\beta)}{w} = 0, \quad (10)$$

or, equivalently,

$$\Sigma\colon\ \frac{a(b^2-c^2)}{u}+\frac{b(c^2-a^2)}{v}+\frac{c(a^2-b^2)}{w}=0, \tag{11}$$

where $[u,v,w]$ *is a tangent to the parabola.*

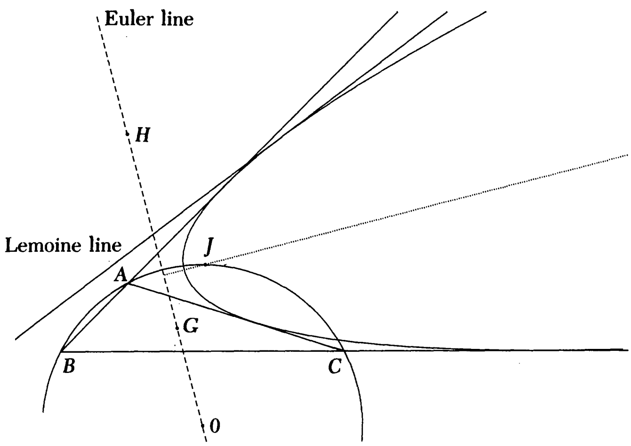

FIGURE 6

Proof. Since the triangles ABC and $A'B'C'$ are perspective from the point P, they are (by Desargues's theorem) perspective from a line. This line has the trilinear representation

$$l=\left[\frac{1}{\sin\beta\sin\gamma+\sin\alpha\sin 2\phi},\frac{1}{\sin\gamma\sin\alpha+\sin\beta\sin 2\phi},\frac{1}{\sin\alpha\sin\beta+\sin\gamma\sin 2\phi}\right], \tag{12}$$

see proof of THEOREM 1, or, equivalently,

$$u:v:w=\frac{1}{\sin\beta\sin\gamma+\sin\alpha\sin 2\phi}:\frac{1}{\sin\gamma\sin\alpha+\sin\beta\sin 2\phi}:\frac{1}{\sin\alpha\sin\beta+\sin\gamma\sin 2\phi}. \tag{13}$$

We now have

$$\begin{aligned}&u(\sin\beta\sin\gamma+\sin\alpha\sin 2\phi)\\&\quad=v(\sin\gamma\sin\alpha+\sin\beta\sin 2\phi)\\&\quad=w(\sin\alpha\sin\beta+\sin\gamma\sin 2\phi),\end{aligned}$$

hence

$$\sin 2\phi=\frac{(v\sin\alpha-u\sin\beta)\sin\gamma}{u\sin\alpha-v\sin\beta}=\frac{(w\sin\alpha-u\sin\gamma)\sin\beta}{u\sin\alpha-w\sin\gamma}, \tag{14}$$

from which we obtain the desired result.

It is obvious that the envelope (11) represents a parabola since the line at infinity $[a,b,c]$ is one of its tangents. Furthermore, this conic (*Kiepert's parabola*) is inscribed in the triangle ABC and has for a fifth tangent the Lemoine line

$x/a + y/b + z/c = 0$. See the important work of Jean Baptiste Joseph Neuberg (Luxembourg 1840–Liège 1926, Professor of Geometry at the Athénée and Lecturer at the École des Mines of Liège) [26, 27, 28]. Using the fact that $a = 2R \sin \alpha$, for example, where R denotes the circumradius of ABC, it follows, from (12), that the line at infinity corresponds to the case $\phi = 0$. However, it appears that this is the only one of the five given tangents that can be obtained by a specific value of ϕ. In addition, the presence of the term $\sin 2\phi$ indicates that, as ϕ varies from 0 to $\pi/2$, one part of the parabola is traversed twice while the other part is not obtained. According to Neuberg, this parabola was first studied in 1884 by the Senior Teacher at the Gymnasium of Recklinghausen, Germany, August Artzt, in a so-called "school programm" [1].

An alternate approach will lead to a projective derivation of the parabola Σ. Consider now a triangle $A^\star B^\star C^\star$ homothetic with the tangential triangle $A_t B_t C_t$ with respect to the circumcentre of ABC as shown in FIGURE 7. Since the tangents to the circumcircle at the vertices have the line coordinates $[0, c, b], [c, 0, a], [b, a, 0]$, the vertices of the tangential triangle have the trilinear representation $A_t = (-a, b, c)$, $B_t = (a, -b, c)$, $C_t = (a, b, -c)$. Recall that the coordinates of the circumcentre O are $(\cos \alpha, \cos \beta, \cos \gamma)$, thus, with respect to a real parameter μ, an arbitrary point A^* has the coordinates

$$A^* = (\mu \cos^2 \alpha - a^2, \mu \cos \alpha \cos \beta + ab, \mu \cos \alpha \cos \gamma + ac).$$

Now we compute the line coordinates of the parallels through A^* to the corresponding tangents and obtain for the remaining vertices under consideration

$$B^* = (\mu \cos \beta \cos \alpha + ba, \mu \cos^2 \beta - b^2, \mu \cos \beta \cos \gamma + bc),$$
$$C^* = (\mu \cos \gamma \cos \alpha + ca, \mu \cos \gamma \cos \beta + cb, \mu \cos^2 \gamma - c^2).$$

Since the lines AA^*, BB^*, CC^* concur at the point

$$((\mu \cos \beta \cos \gamma + bc)^{-1}, (\mu \cos \gamma \cos \alpha + ca)^{-1}, (\mu \cos \beta \cos \alpha + ba)^{-1}),$$

the triangles ABC and A^*, B^*, C^* are perspective from a line, i.e., the points $N_a = BC \cap B^\star C^\star$, $N_b = CA \cap C^\star A^\star$, $N_c = AB \cap A^\star B^\star$ belong to one line that we denote by $N_a N_b N_c$ in the sequel. The assignment $N_a \mapsto C^\star$ is a perspectivity from the line BC to the line OC_t, and the assignment $C^\star \mapsto N_b$ is another perspectivity; hence the composite assignment is a projectivity and so, the lines $N_a N_b N_c$ envelope a conic; see [36, pp. 109].

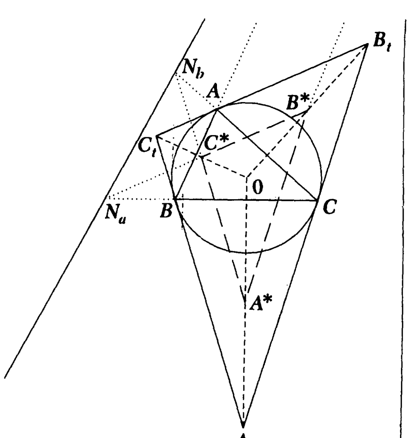

FIGURE 7

To see that this conic is actually Kiepert's parabola, we show how to identify the five specific tangents listed above. The Lemoine line is an obvious tangent since it is just the axis of perspectivity of the triangles ABC and $A_tB_tC_t$. Next, consider the case when $A, A^\star, C^\star$ are collinear, see FIGURE 8. Now $C^\star A^\star \cap CA = N_b = A$ and $A^\star B^\star \cap AB = N_c$ is always a point on AB, thus the side AB is a tangent to the conic. Similar arguments show that BC and CA are tangents also. Finally, when $A^\star, B^\star, C^\star$ are themselves on the line at infinity, i.e., $\mu = -4R^2$, the line $N_aN_bN_c$ is the tangent at infinity.

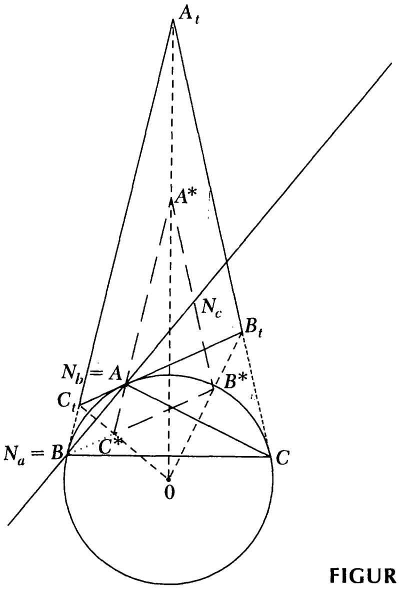

FIGURE 8

As in the case of the hyperbola, we list some properties of the parabola that serve to demonstrate that this conic also plays an important role in the geometry of the triangle.

(i) The Euler line, see [18], of the triangle ABC is the directrix of Kiepert's parabola. To prove this, we shall need the following, see [31, p. 70].

LEMMA. *The directrices of all parabolas inscribed in a triangle pass through the orthocentre.*

Since this reference may not be well known today, we sketch a proof. The foot of the perpendicular from the focus of a parabola to any tangent belongs to the tangent at the vertex. Thus, the three feet of the perpendiculars from the focus to the sides of a tangential triangle are collinear showing that the focus belongs to the circumcircle of the triangle and that the tangent at the vertex is its Wallace-Simson line. The latter bisects the segment joining the focus to the orthocentre and, consequently, the orthocentre belongs to the directrix, see also [31, pp. 48ff].

To see (i), first note that the above lemma implies that the directrix of Kiepert's parabola Σ contains the orthocentre H of the triangle ABC. Second, consider the tangent to Σ that corresponds to $\phi = -\gamma$ where A' is on AC and B' is on BC. Then, the circumcentre O of ABC is the orthocentre of $A'B'C$ and Σ is also inscribed in this triangle. Now, by the lemma, the point O belongs to the directrix of Σ. The directrix thus contains the points O and H and hence, is the Euler line e of ABC.

(ii) The coordinates of the focus J of Kiepert's parabola are given by

$$J = \left(\frac{1}{\sin(\beta - \gamma)}, \frac{1}{\sin(\gamma - \alpha)}, \frac{1}{\sin(\alpha - \beta)} \right).$$

To see this note that the pole of a line $[u, v, w]$ with respect to Σ is given by

$$\begin{aligned}&\big[(v \sin^2 \gamma \sin(\alpha-\beta) + w \sin^2 \beta \sin(\gamma-\alpha)),\\&(w \sin^2 \alpha \sin(\beta-\gamma) + u \sin^2 \gamma \sin(\alpha-\beta)),\\&(u \sin^2 \beta \sin(\gamma-\alpha) + v \sin^2 \alpha \sin(\beta-\gamma))\big].\end{aligned}$$

Since $e = [\sin 2\alpha \sin(\beta-\gamma), \sin 2\beta \sin(\gamma-\alpha), \sin 2\gamma \sin(\alpha-\beta)]$, the coordinates of J are the claimed ones.

In the sketch of the proof of the lemma above it has been mentioned that the focus of any parabola inscribed a triangle belongs to the circumcircle of this triangle. In our case, it's an easy exercise to verify that the given coordinates for J satisfy the equation

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

which is that of the circumcircle of ABC.

(iii) If a conic is inscribed in a triangle, then the joins of the vertices of this triangle and the points of contact are concurrent in what may be termed the *Brianchon point* of the conic with respect to the triangle, see [**36**, p. 111]. Since a conic inscribed in the triangle ABC has an equation of the form $f/u + g/v + h/w = 0$, its Brianchon point is easily seen to have coordinates $(1/f, 1/g, 1/h)$. Thus, for Kiepert's parabola, it is, from (11), $(1/a(b^2-c^2), 1/b(c^2-a^2), 1/c(a^2-b^2))$, which is the *Steiner point* S of the triangle, see [**18**]. This is the point of concurrence of the three lines drawn through the vertices of a triangle parallel to the corresponding sides of Brocard's first triangle. In addition, the Steiner point is on the circumcircle of ABC, see [**14**, pp. 281ff] and FIGURE 9.

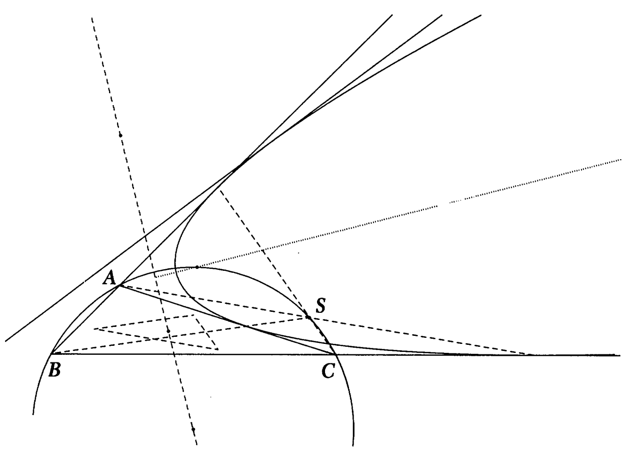

FIGURE 9

(iv) On the basis of property (xi) of the hyperbola, the point form of the parabola has equation

$$\Sigma: f^2x^2 + g^2y^2 + h^2z^2 - 2fgxy - 2ghyz - 2hfzx = 0,$$

Artzt also studied other parabolas associated with the triangle. Of particular relevance at this time is a trio referred to by Casey as the Artzt's parabolas (second group). Consider the configuration of THEOREM 1. Since the line $A'B'$, for example, is the join of two projectively related points, it envelopes a conic. This conic is a parabola such that the internal and external bisectors of $\measuredangle BCA$ are tangents as are the perpendicular bisectors of BC and CA. Similar arguments hold for the lines $B'C'$ and $C'A'$.

5. Results Not Found in the Available Literature

Here we present some material, in the form of theorems, which we believe to be new.

THEOREM 3. *The centre of the circle inscribed in the triangle DEF, where D, E, F are the midpoints of the sides BC, CA, AB respectively of the given triangle ABC, lies on Kiepert's hyperbola.*

Proof. Since the triangle DEF is homothetic to the triangle ABC with factor $-1/2$, the radius ρ of the given circle, also known as the *Spieker circle*, see [**14**, p. 226], is $r/2 = \Delta/2s$, where $s = (a + b + c)/2$ and Δ denotes the area of the triangle ABC. Consequently, the distance d_a of the centre V of the Spieker circle from the side BC of the triangle ABC is given by the equation

$$d_a = \frac{h_a}{2} - \rho = \Delta\left(\frac{1}{a} - \frac{1}{2s}\right),$$

where h_a denotes the altitude to the side BC. The coordinates of V are now easily seen to be

$$V = \left(\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}\right),$$

which satisfy the equation (5) of Kiepert's hyperbola, see FIGURE 10.

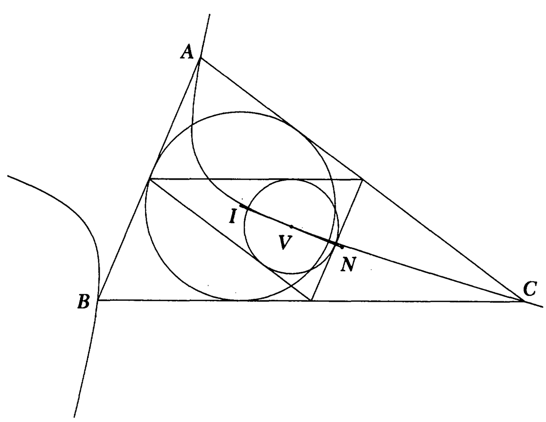

FIGURE 10

Remark. By means of barycentric coordinates one can show that V is midway between the incentre I and the Nagel point N, see Section 2b. For further properties of the Spieker circle see [**14**, pp. 226ff] and [**18**].

In M'Cay [**23**], it is given that the point D, the centre of homology of the triangle ABC and Brocard's first triangle, lies on the hyperbola. It is, by its very definition, nothing but the point P of concurrence in the sense of our THEOREM 1 corresponding to $\phi = -\omega$. We have been able to link this result with the following already mentioned fact:

THEOREM 4. *Brocard's third point lies on Kiepert's hyperbola.*

Proof. The barycentric coordinates of this point, which we denote as τ_3, are given in [**5**, p. 66] as $(1/a^2, 1/b^2, 1/c^2)$, which implies that the trilinear coordinates are $\tau_3 = (1/a^3, 1/b^3, 1/c^3)$. It now becomes a trivial exercise to verify that these satisfy equation (5).

THEOREM 5. *The points D and τ_3 are one and the same.*

Proof. From (6), the coordinates of the point on Kiepert's hyperbola corresponding to $\phi = -\omega$ are

$$\left(\frac{1}{\sin(\alpha-\omega)}, \frac{1}{\sin(\beta-\omega)}, \frac{1}{\sin(\gamma-\omega)}\right),$$

which can also be written in the form

$$\left(\frac{1}{(\sin\alpha\cot\omega-\cos\alpha)}, \frac{1}{(\sin\beta\cot\omega-\cos\beta)}, \frac{1}{(\sin\gamma\cot\omega-\cos\gamma)}\right).$$

But $\cot\omega=(a^2+b^2+c^2)/4\Delta$, see [14, pp. 264ff], and $4\Delta = 2bc\sin\alpha$, thus $(1/\sin\omega(\sin\alpha\cot\omega-\cos\alpha))=(bc/a^2\sin\omega)$ and similarly for the other two coordinates. We now have

$$\left(\frac{bc}{a^2\sin\omega}, \frac{ca}{b^2\sin\omega}, \frac{ab}{c^2\sin\omega}\right)=\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right)$$

as the coordinates of τ_3.

We actually discovered that τ_3 was on the hyperbola before seeing the information in Casey. Since we believe that this was accomplished by a rather pretty argument, we supply some details. By comparing equations (5) and (11) it is easy to see that the two are related by the elliptic polarity $\rho x_i = \Sigma_j a_{ij} u_j$; $i, j = 1,2,3$; $\rho \neq 0$, where $a_{11}=b^2c^2$, $a_{22}=c^2a^2$, $a_{33}=a^2b^2$ and $a_{ij}=0$ when $i \neq j$, which maps the points of the hyperbola to the tangent lines of the parabola. Brocard's third point corresponds to the Lemoine line $[bc, ca, ab]$ under this transformation. We note further that $\tau_3 = K^t$, the isotomic conjugate of the symmedian point. The reader may wish to use this idea to find other meaningful points and lines associated with these conics.

As an aside, we have derived a further result with respect to the point τ_3.

THEOREM 6. *Brocard's third point is collinear with the centre of the Spieker circle and the isotomic conjugate of the incentre.*

Proof. Since the determinant

$$\begin{vmatrix} \dfrac{b+c}{a} & \dfrac{c+a}{b} & \dfrac{a+b}{c} \\ \dfrac{1}{a^3} & \dfrac{1}{b^3} & \dfrac{1}{c^3} \\ \dfrac{1}{a^2} & \dfrac{1}{b^2} & \dfrac{1}{c^2} \end{vmatrix}$$

vanishes, the result follows.

Remark. The barycentric coordinates of Brocard's first and second points are $\tau_1=(1/b^2, 1/c^2, 1/a^2)$ and $\tau_2=(1/c^2, 1/a^2, 1/b^2)$, so that the barycentric coordinates of $\tau_3=(1/a^2, 1/b^2, 1/c^2)$ complete the cyclic order. This may be the reason for the name *Brocard's third point*, which we only found in Casey [**5**, p. 66], in the coordinate form above, with no further information given. Kimberling [**18**] lists this point as just one of 91 *polynomial centers* of the reference triangle and mentions our THEOREM 6 in a slightly different form.

6. Conclusion

Even now there are other aspects of these conics that we have not touched upon as they seem to require a more thorough knowledge of the geometry of the triangle than that of the general reader. However, what is included should serve to convince the reader that Kiepert's hyperbola and Kiepert's parabola are not only interesting in their own right, but also, they constitute an important chapter of the geometry of the triangle. In FIGURE 11 we show them together for the first time. The reference triangle is deliberately chosen to be right-angled since the hyperbola is best illustrated with respect to an acute triangle while, in the case of the parabola, the obtuse case is more convenient.

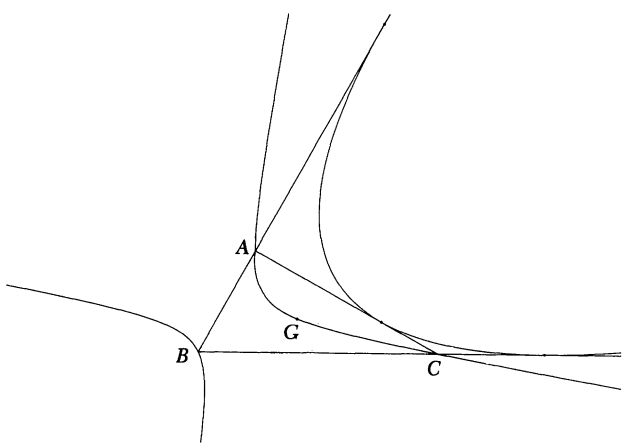

FIGURE 11

Acknowledgements The authors would like to thank the following:

- One of the referees for suggesting the present proof that the points N_a, N_b, N_c are collinear; this is much nicer than our original proof. Also, to the same referee, for a shortcut of our original argument concerning the projective derivation of the parabola.
- Mrs. B. Eddy for locating many of the references, some of which were not readily obtainable.

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"The eagle putt was long, and at that moment it seemed almost as long as the Iberian Peninsula that spawned the man who stood over it at the 15th hole of Augusta National today. The Masters is won and lost on such putts, and José Maria Olazabal added his name to the list of men who have accepted the challenge...

... He learned early the value of imagination around the greens, and it was that imagination that carried him to victory. On a day when the firm and fast putting surfaces were as difficult to solve as linear equations, Olazabal spent much of the day doing an impression of his more famous countryman, Seve Ballesteros, when it came to getting up and down..."

--Larry Dorman, *New York Times*, April 11, 1994, C1
(sent by Robert A. Russell, New York)

NOTES

Museum Exhibits for the Conics

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It is rather common nowadays that museums, especially those of the "science center" type, have showcases where visitors can perform experiments. In the Futures Museum in Borlänge we have made an exhibit where interested persons can construct parabolas. In this note we describe this showcase. Analogous exhibits can be made for constructing ellipses and hyperbolas. Although the ideas that underlie the constructions have been described by others (three references below), they deserve to be better known.

An exhibit for parabolas On a table with a "whiteboard surface" for drawing, a straightedge is fastened. To one side of this straightedge, some holes drilled in the tabletop can be filled with a nail or peg. A draftman's triangle (or carpenter's square) is placed so that a leg touches the peg and the vertex of the right angle touches the straightedge (FIGURE 1)*. A straight line is drawn along the other leg of the triangle (using a water-based marker). This is repeated for several positions of the triangle, as the vertex of the triangle travels along the straightedge. The result is a set of lines in the envelope of tangents to a parabola (FIGURE 2). The peg is the focus of the parabola and the straightedge is tangent to the vertex of the parabola. The drawn lines can be washed off and a new parabola constructed for another position of the peg.

The right triangle used to draw the envelope of tangents to the parabola makes practical use of the following geometric property:

The intersection of a tangent to a parabola with the perpendicular to it from the focus lies on the line tangent to the vertex of the parabola.

This can be proved in a straightforward manner by introducing a coordinate system so that the equation of the parabola is $y^2 = 2px$. It is easily shown that the equation of a tangent line to this parabola is of the form $y = kx + p/(2k)$, and the perpendicular to that line from the focus $(p/2, 0)$ has equation $y = -x/k + p/(2k)$. Thus both lines pass through the point $(0, p/(2k))$. This proof can be found in analytic geometry books that detail the properties of conic sections; for example, see [1], pp. 111–112.

An exhibit for ellipses The usual construction of an ellipse uses the property that the sum of the distances from a point on the ellipse to the two foci is a constant. But a variation on the construction of the parabola given above will yield an ellipse. In this

*All figures were drawn using the computer program *The Geometer's Sketchpad*.

construction, the fixed straightedge is replaced with a large fixed circular ring with a low rim, and the drilled holes in which a peg can be placed are inside the circle. As before, a right triangle is positioned with one leg against the peg with the vertex of the right angle on the circle (FIGURE 3). A line is drawn along the other leg of the triangle. As the vertex of the triangle moves around the circle, the series of lines that are drawn form part of the envelope of tangents to an ellipse (FIGURE 4). The peg is one of the two foci of the ellipse.

This construction depends on the following geometric property:

The intersection of a tangent to an ellipse with the perpendicular to it from a focus lies on a circle which has the major axis of the ellipse as a diameter.

The proof of the property is analogous to that for the parabola; see [1], pp. 139–140.

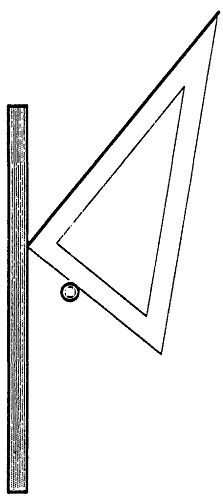

FIGURE 1

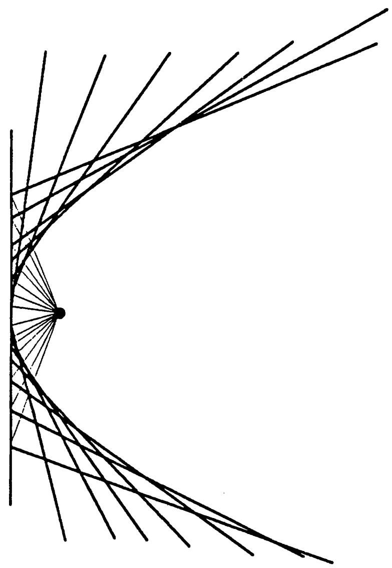

FIGURE 2

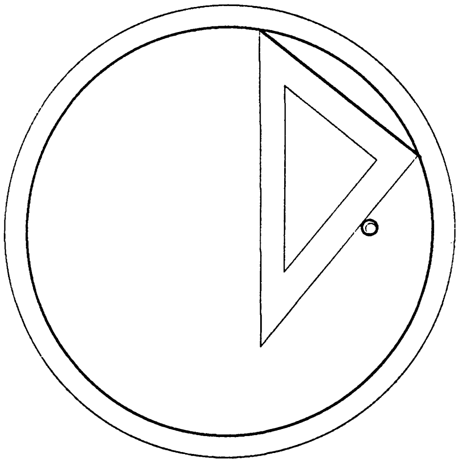

FIGURE 3

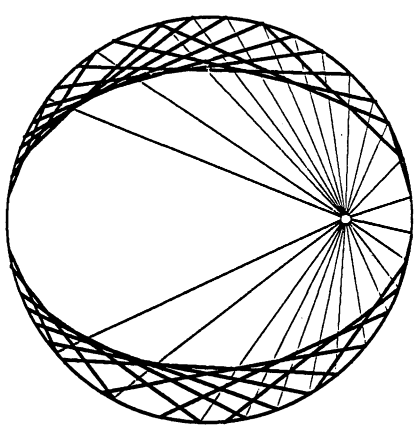

FIGURE 4

An exhibit for hyperbolas For this exhibit, a circular plate is fastened to the table, with holes (for pegs) drilled outside the circle, on a line through the center of the circle and equidistant from that center. As before, the right triangle is placed with one leg against a peg, vertex of the right angle against the circle, and a line is drawn along the other leg (FIGURE 5). As the vertex of the triangle is moved to different positions, the series of lines drawn on one side of the circle enclose one branch of a hyperbola. The other branch is outlined by tangents by repeating the procedure with a peg placed at the symmetric position on the other side of the circle (FIGURE 6). The two pegs are foci of the hyperbola and the circle is tangent to the two vertices of the hyperbola.

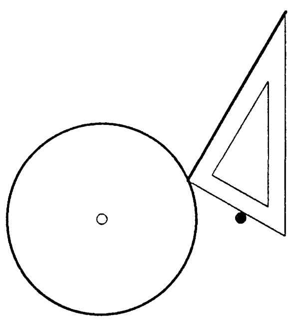

FIGURE 5

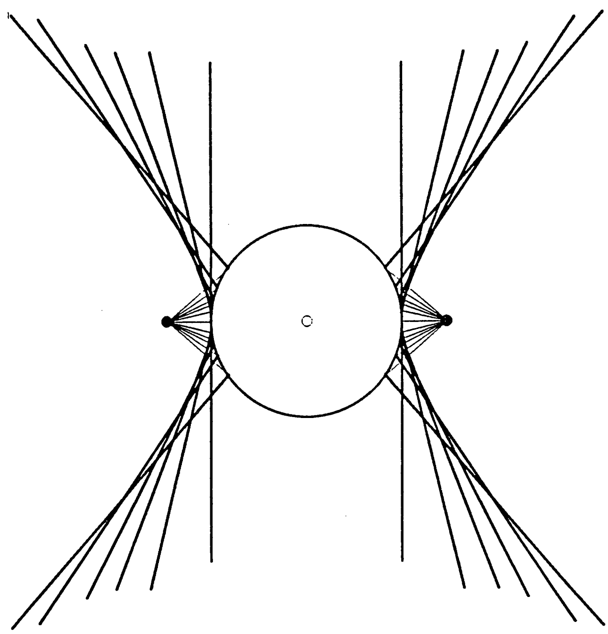

FIGURE 6

The construction uses this geometric property:

The intersection of a tangent to a hyperbola with the perpendicular to it from a focus lies on the circle that has the transversal axis of the hyperbola as a diameter.

The proof of this fact is an immediate consequence of the argument for the ellipse, gotten just by changing the sign in the equation for the curve (see [1], pp. 179–180).

These constructions of the three conics in the manner described is nicely implemented by computer programs that do geometric constructions or accurate drawings of circles and perpendicular lines. This also suggests interesting "showcase" exhibits to involve visitors using an interactive computer program. In fact, using such a program, both branches of the hyperbola can be constructed using just one focus. This is because in general, a line L through a focus of the hyperbola has two points of intersection with the circle described on the transverse axis. At each of these points, a line can be constructed perpendicular to L; each of these perpendiculars is a tangent to the hyperbola (FIGURE 7). A second construction of the hyperbola in FIGURE 6, this time using just one focus, is shown in FIGURE 8.

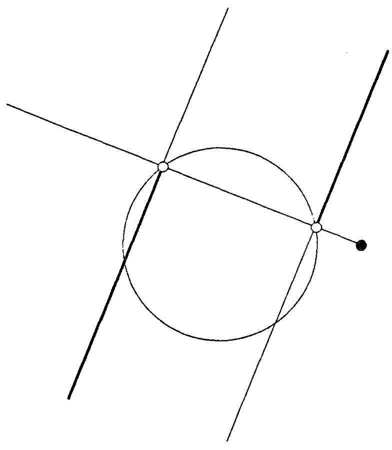

FIGURE 7

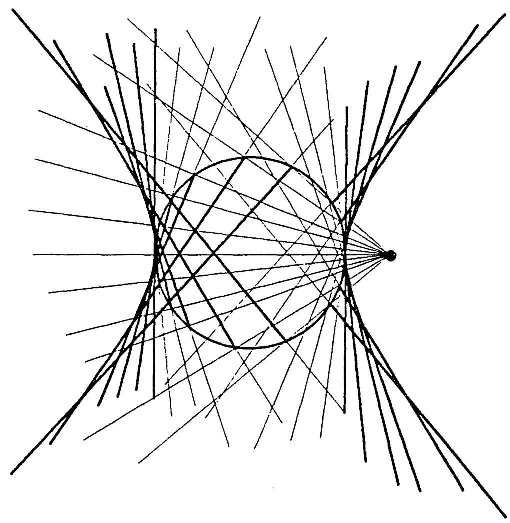

FIGURE 8

Acknowledgement We acknowledge important suggestions by Doris Schattschneider, who also helped us find the references.

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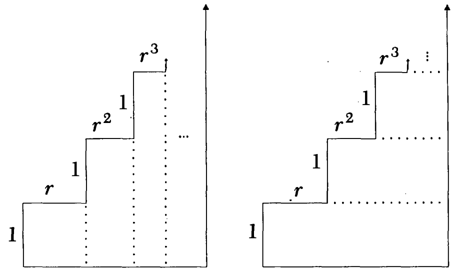
2. Alfred S. Posamentier and Jay Stapelman, *Teaching Secondary School Mathematics*, 3rd edition, 1990, pp. 447–450.
3. Robert C. Yates, *Curves and Their Properties*, National Council of Teachers of Mathematics, Reston, VA, 1974. Reprint of 1952 edition, p. 50.

Proof Without Words: Gabriel's Staircase

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \quad \text{for} \quad 0<r<1$$



$$\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} r^i = \frac{r}{(1-r)^2}$$

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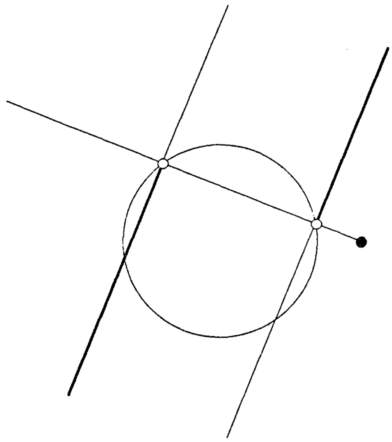
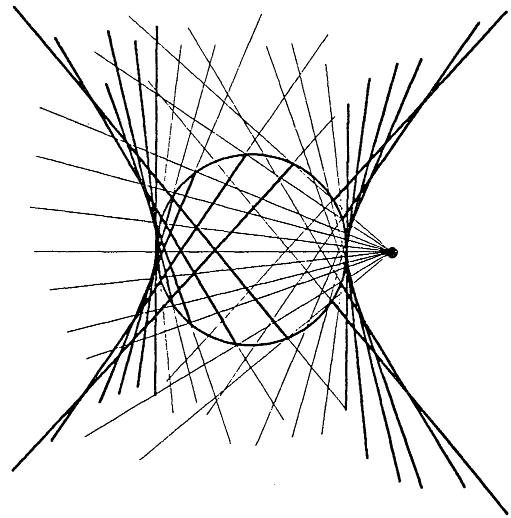


FIGURE 7



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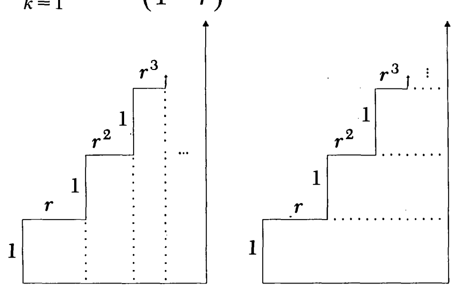

$$\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} r^i = \frac{r}{(1-r)^2}$$

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Mortality for Sets of 2×2 Matrices

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Do undergraduate mathematics majors know much about what has been discovered in mathematics during this century? One such concept, blossoming in the 1930s, is the theory of solvability. A problem is said to be *solvable* if there exists a decision procedure, or algorithm (i.e., a defined procedure for solving a problem using a finite number of steps), that will yield a solution to the problem. The purpose of this paper is to give some examples of open solvability problems and to demonstrate the solution to another. This material is accessible to a first-semester linear algebra student.

To more easily understand some of the concepts to follow, consider the following set of 2×2 matrices:

$$A=\left\{\begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&2\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right\}.$$

Note that the product, $\begin{pmatrix}6&0\\0&6\end{pmatrix}$, of the first two matrices of this set has equal elements along the diagonal. Also, the product of the third matrix and itself has the same property. In general, a finite set of matrices is said to have an *equalizing product* if there exists a product of (*any number of*) matrices from this set having equal entries in certain pre-specified positions. If we ask that the equal positions be those on the main diagonal, the set A has an equalizing product, but the set

$$B=\left\{\begin{pmatrix}2&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&4\end{pmatrix},\begin{pmatrix}1&0\\0&2\end{pmatrix}\right\}$$

has no equalizing product since the upper left entry of any product will always be less than the lower right entry. Note that equalizing products can include single matrix factors (i.e., the set of matrices under consideration may contain a member with equal entries already in the specified positions), and they can also include the same matrix more than once as a factor. It is this latter case that makes finding equalizing products in a given set of matrices more challenging, since any one matrix can be used as a factor an arbitrarily large number of times. In fact, if we wanted an algorithm for deciding whether or not an equalizing product existed for a given set of matrices, a procedure that "checked every possible product" of matrices in the set would never give us the answer if the set does not have an equalizing product, since the process would never terminate.

Suppose we are considering a particular class of matrices, for example, all 2×2 diagonal matrices with integer entries. Is there some algorithm that will tell whether or not *any* finite set of such matrices has an equalizing product? It turns out that the answer to this *equality of entry problem* is "Yes!" (See **[1]**.) That means this particular problem is solvable.

To understand more about equality of entry problems, we must first grasp the ideas behind another related problem. A nonempty finite set of $n \times n$ nonzero matrices with integer entries is said to be *mortal* if there exists some product of matrices in the set that is equal to the zero matrix. The *mortality problem* is the question of whether or not there exists an algorithm for deciding mortality for any given set of matrices. If no such algorithm exists we say that the mortality problem is *unsolvable*.

It has been shown (see [2], [3]) that for $n \geq 3$, the mortality problem is equivalent to a previously known unsolvable problem and is therefore unsolvable itself. For $n = 1$, the fact that the mortality problem is solvable is trivial. Therefore, the only case that is still unresolved is the 2×2 case.

Certain special cases of the 2×2 mortality problem have been shown to be solvable. For example, if the matrices are all upper or all lower triangular, we need only check to see if there exists, for each diagonal position, at least one matrix in the given set with a zero in that position. See also [1]. Although the general case may be unsolvable, it may be possible to find one level of restrictions (as above) that makes the problem solvable, and by easing the restrictions by some small amount, find the problem suddenly unsolvable!

Krom and Krom [1] showed that the following equality of entry problem is equivalent to the 2×2 mortality problem (i.e., if one is solvable, so is the other):

(1) Does there exist an algorithm that decides for any finite set P of nonsingular 2×2 matrices with integer entries whether there is a product of members of P that is equal to a matrix C with $c_{21} = c_{22}$?

These same authors then proposed the following subproblem of (1):

(2) Does there exist an algorithm that decides for any finite set P of nonsingular lower triangular 2×2 matrices with integer entries whether there is an equalizing product of members of P such that the lower two entries of the product are equal?

If (2) can be answered in the negative, then so, too, can problem (1), and therefore, the mortality problem, is unsolvable.

Again, if problem (1), or even problem (2), is unsolvable, perhaps there are levels of restrictions we can add so that passage from one level to another might lead us from unsolvability to solvability. To that end, we have imposed additional restrictions on (2) to arrive at a case we can show is solvable:

> Let P be a finite set of nonsingular lower triangular 2×2 matrices with integer entries. Let the entries in each particular matrix have the same sign, say nonnegative, and let the upper left entry be at least as large as the lower right entry. (We can assume that all of the entries are nonnegative, since multiplying by -1 will not change the existence of an equalizing product.) In other words, we are working with the set
>
> $$\left\{ \begin{pmatrix} m_{11} & 0 \\ m_{21} & m_{22} \end{pmatrix} : m_{ij} \in \mathbb{Z},\ m_{ij} \geq 0,\ m_{11} \geq m_{22}, \text{ and } m_{11}m_{22} \neq 0 \right\}.$$

Before demonstrating that an algorithm exists for deciding this restrictive case, note that the point is to show that this case is solvable. All we must do is provide *an* algorithm, no matter how inefficient it may be.

Demonstration of algorithm. Let P be a finite set of n nonsingular lower triangular 2×2 matrices $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$, where x, y, and z are nonnegative integers, and $x \geq z$. The goal is to show that there exists an algorithm that decides whether there is a product formed with members of P that is equal to a matrix E such that $e_{21} = e_{22}$.

Since multiplying a matrix of P by a scalar does not alter the outcome of the equality of entry problem, we can multiply each matrix by its respective $1/z$. (Remember that the members of P are nonsingular.) This gives us a set P' of matrices of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, where $a \geq 1$.

Note that when two such matrices are multiplied,

$$\begin{pmatrix} a_1 & 0 \\ b_1 & 1 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ b_2 & 1 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 \\ a_2b_1 + b_2 & 1 \end{pmatrix},$$

so

$$e_{21} = a_2b_1 + b_2, \text{ and } e_{22} = 1.$$

If we multiply three of these matrices, we get

$$\begin{aligned} e_{21} &= a_3a_2b_1 + a_3b_2 + b_3 \\ &= a_3(a_2b_1 + b_2) + b_3, \text{ and} \\ e_{22} &= 1. \end{aligned}$$

In general, if there are m matrices in the product, then

$$e_{21} = a_m x_m + b_m \quad \text{and} \quad e_{22} = 1,$$

where

$$x_1 = 0 \quad \text{and} \quad x_{m+1} = a_m x_m + b_m.$$

Note that as factors are adjoined to the right of the product, x_m is nondecreasing (since the a's are at least 1 and the b's are nonnegative).

We would like to know whether or not we have a collection of $\begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix} \in P'$, where $i = 1, \dots, m$, such that

$$a_m x_m + b_m = 1,$$

or equivalently,

$$x_m = \frac{1 - b_m}{a_m}.$$

Note that the right-hand side of the last equation involves only components from the last (or mth) matrix in the product, and the left-hand side involves only components from the first $m-1$ matrices in the product. This tells us, for example, that if the mth matrix does not have $b_m = 1$, then the corresponding nondecreasing $x_m > 0$ and, therefore, at least one of the previous $m-1$ matrices in an equalizing product is not a diagonal matrix, since some $b_i > 0$ for $i < m$.

To test a finite set P of matrices for an equalizing product, we first form the new set P', then assign an ordering to P', and then test each matrix of P' in that order to see if it could be the last (or mth) matrix in an equalizing product of m matrices. The steps of the algorithm are as follows:

1. If all of the matrices in P' are diagonal, then stop the algorithm with the answer "no" (i.e., there is no equalizing product).

   Else, set $i = 0$.
2. Set $i = i + 1$, $m = 0$.
3. If the ith matrix is $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, set $d = (1 - b/a)$.
4. If $d = 0$, then stop the algorithm with the answer "yes" (i.e., the ith matrix in P is an equalizing product). If $d < 0$, then go to step 2.
5. Set $m = m + 1$.
6. Consider the ith matrix in P' to be the mth factor in a product of m matrices. If $m = 1$, go to step 5.

7. Run through the n^{m-1} options for what the first $m-1$ matrices in the product could be, calculate the corresponding x_m, and make a record of all of these x_m calculated.
8. Case: There exists an $x_m = d$.
   Stop the algorithm with the answer "yes".
   Case: For all nonzero x_m, $x_m > d$.
   If $i < n$, go to step 2.
   Else, stop the algorithm with the answer "no", since all the entries of the matrices are nonnegative and the x_m are nondecreasing.
   Otherwise: Go to step 5.

To illustrate how this algorithm works, two examples are provided.

Example 1. Let

$$P = \left\{ \begin{pmatrix} 9 & 0 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 8 & 6 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 9 & 4 \end{pmatrix} \right\}.$$

The first matrix in P is the only one that has a chance of being the last factor in an equalizing product. The other three each have $d < 0$. The d for the first matrix, though, is less than all of the possible x_3, and, therefore, this set P cannot produce an equalizing product.

Example 2. Let

$$P = \left\{ \begin{pmatrix} 9 & 0 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 1 & 5 \end{pmatrix} \right\}.$$

The first matrix gets ruled out after calculating the x_3's. The second matrix again has $d < 0$. During the calculation of the x_2's for the third matrix, it is discovered that squaring that matrix is an equalizing product. Notice that this algorithm does not find all the equalizing products. For example, the third matrix times the fourth is also an equalizing product.

The reader may wish to examine other equality of entry forms, such as products of 2×2 matrices having entries that agree in the second column's two positions. Also, for information on the unsolvability of a different type of 2×2 matrix situation, see [4].

The author is indebted to the editor and anonymous referees for helpful comments.

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Formal Justice and Functional Equations

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1. Introduction The requirement of formal justice leads to fundamental problems in sociology. For example, we mention the wage of an individual based on certain qualifications, or, on the negative side, the punishment of an individual based on a crime committed by that individual. We will call treatments based on such (positive or negative) qualifications *compensations*, and refer to the corresponding qualifications as *compensable properties*.

Loosely speaking, formal justice requires that there should be an adequate relationship between compensable property and compensation. In order to give a mathematical definition of such a relationship, it is first necessary to measure both compensation and compensable property on a numerical scale. This is rather straightforward for compensation (wages are measured in monetary units), but a difficult endeavor for compensable properties. *A priori*, the number of years of education of an individual, or the number of years of experience in a certain profession or trade look like good measures of these compensable properties; whether they give realistic measurements of the degree of education or the experience is questionable.

Nevertheless, measurement theory attempts to assign positive real numbers in some way to persons, objects or properties: the compensable properties, P, described above. Whether such assignments can give meaningful measurements depends on the property under consideration, and is, in any case, not a mathematical question. For a more detailed discussion of this problem we refer the reader to [1], where the principles of formal justice and the difficulties of measurement are considered from a sociological point of view. There, an assignment $P \to \mathfrak{R}_+$ is called a *measurement system* if it is (a) *reliable*, i.e. invariant with respect to the user of the system, and (b) *accurate*, i.e. if relations between objects are reflected by relations between their assigned compensation.

This is clearly not a rigorous mathematical definition, due to the remaining fuzziness of certain terms. To get a little closer to reality, we now focus on wage systems, which are measurement systems assigning wages to persons based on measured compensable properties of these individuals. We call a wage system *formally just* if

i) both the concerned set of people and the set of possible wages are recognized as relational systems,

ii) the wage system is reliable,

iii) the wage system is accurate, and

iv) the wage system possesses an accurate inverse in the sense that to any two different wages there are well-defined different "prototypes" of people who qualify for these wages.

Following [1], we suggest that (iii) means that the wage system is a homomorphism from the system of (prototypes of) persons to the positive real numbers (expressive homomorphic requirement); (iv) means that this homomorphism is actually an isomorphism (justificatory homomorphic requirement). Soltan [1] refers to (iii) and (iv) together as the *first isomorphic requirement of formal justice*.

We emphasize again that this definition remains nonmathematical. In order to attain real rigor, one has to define the relations to be used on both the set of concerned individuals (or on the measurements of their compensable properties) and on the set of possible wages. This is our objective in the next section.

2. Ratio scales and a first functional equation Little mathematics can be done in the generality that we have allowed so far. We now consider the case of one compensable property and assume that both the compensable property and the wage are measured on a *ratio scale*; more precisely, we assume that the property is measured by real nonnegative numbers $x \geqslant 0$; a prototype possessing the property, by definition, of degree 1 has been chosen, and the relation of all other prototypes to this one is defined to be the quotient $x/1 = x$. The relation between people measured as $x > 0$ and as $y > 0$ is then the quotient x/y.

The wage system will now be a function $m\colon \Re_+ \to \Re_+$, defined on the (measured) values of the compensable property. If the relation between any two wages is also given by their ratios and if, for the present, we assume in addition that the prototype measuring in at 1 gets a salary also measured at 1, then the first isomorphic requirement insures that m must satisfy the functional equation

$$m\left(\frac{x}{y}\right) = \frac{m(x)}{m(y)} \tag{1}$$

(i.e., the ratio between the measured compensable properties is mapped to the ratio between their assigned wages).

This definition leads to several conceptual problems. Before we address these, we consider those functions $m\colon \Re_+ \to \Re_+$ that satisfy (1). Indeed, equation (1) is well known (the transformation $g(t) := \ln m(e^t)$ leads to the even more familiar equation $g(t+s) = g(t) + g(s)$), and we have:

THEOREM 1. *The only functions* $m\colon \Re_+ \to \Re_+$ *that satisfy* (1) *and are continuous at some point are of the form*

$$m(x) = x^p \qquad (p \in \Re). \tag{2}$$

Under the continuity assumption made here, the assertion is only an exercise in real analysis. We mention that the assertion holds under the much weaker assumption that g is bounded on one side on some set of positive Lebesgue measure (see [2] and [4]). Under no conditions on g at all, the functional equation $g(t+s) = g(t) + g(s)$ possesses discontinuous solutions, see [3] or [5]. In the present context (formal justice), it is natural to restrict the analysis to continuous functions, because discontinuous solutions of $g(t+s) = g(t) + g(s)$ are unbounded on both sides on every set of positive Lebesgue measure.

We discuss the weaknesses of equation (1) as a definition of formal justice.

a) First, there is no rigorous reason why one should use ratios as the appropriate relation for either the comparison of the degree to which individuals possess a compensable property, or for the comparison of their wages. One could consider differences rather than ratios, or one could consider differences on one side and ratios on the other. The use of completely different relations is also conceivable.

Ratio scales are, at least, suggested by tradition. The principle of proportional justice, which goes back to Aristotle, says that

$$\frac{m(x)}{m(y)} = \frac{x}{y}, \tag{3}$$

i.e. $m(x)=c\cdot x$ for some constant c. We shall see that (3) is a more restrictive version of justice than the generalization of (1) that we propose below.

b) A smaller problem is that we have apparently not taken into account market influences. We briefly address this point again in Section 5.

c) The assumption $m(1)=1$, which clearly also follows from equation (1), is problematic since equation (1) is obviously not invariant under scale changes $x'=\varepsilon x$, $m'=\delta m$. We address this problem in the next section.

3. A scale invariant formulation of formal justice Formal justice should in no way depend on whether we measure time in hours or minutes, or whether we measure a salary in dollars or cents. However, equation (1) is not invariant under such changes: If $\tilde{m}(x)=Cm(x)$ and m satisfies (1), $\tilde{m}$ satisfies

$$\tilde{m}\left(\frac{x}{y}\right)=C\frac{\tilde{m}(x)}{\tilde{m}(y)}$$

rather than (1). And if $\overline{m}(x)=m(\varepsilon x)$, we have

$$\overline{m}\left(\frac{x}{y}\right)=m(\varepsilon)\frac{\overline{m}(x)}{\overline{m}(y)}.$$

This lack of scale invariance is of course also evident from the solution $m(x)=x^p$ given by Theorem 1. Inspired by this observation, we now attempt a rigorous definition of formal justice for ratio scales.

Definition and *Corollary. A function* $m\colon \mathfrak{R}_+\to\mathfrak{R}_+$ *is suitable as a formally just wage function if there is a positive constant C such that*

$$m\left(\frac{x}{y}\right)=C\frac{m(x)}{m(y)} \tag{4}$$

for all $x,y>0$. *If such a function is continuous for at least one* $x>0$, *then m is necessarily of the form*

$$m(x)=Cx^p, \tag{5}$$

where $C>0$ *and* $p\in\mathfrak{R}$.

4. *N* compensable properties Things get a lot more interesting if, as in real situations, more than one compensable property is to be considered. If N properties are allowed, measured by $(x_1,\dots,x_N)\in\mathfrak{R}_+^N$, then we say that

$$m\colon \mathfrak{R}_+^N\to\mathfrak{R}_+$$

is suitable as a formally just wage function if m satisfies (4) with respect to each variable (where C and p, of course, can depend on all the others). If m is continuous, then by the corollary

$$\begin{aligned} &m(x_1,\dots,x_i,\dots,x_N)\\ &\quad=C_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)\cdot x_i^{p_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)} \end{aligned} \tag{6}$$

for all $i=1,\dots,N$. We classify all continuous functions satisfying this criterion in

THEOREM 2. *Every continuous function* $m\colon \Re_+^N \to \Re_+$ *that satisfies* (6) *for all* $i = 1, \dots, N$ *is of the form*

$$m(x_1, \dots, x_N) = \exp\left[\sum_{M \subset \{1, \dots, N\}} p_M \prod_{i \in M} \ln x_i\right]. \tag{7}$$

Remarks. $\sum_{M \subset \{1, \dots, N\}}$ denotes summation over all the 2^N subsets of $\{1, \dots, N\}$, and the p_M are real parameters indexed by M. For $N = 2$, (7) yields

$$m(x_1, x_2) = C x_1^{p_1} x_2^{p_2} \exp[p_{1,2} \cdot \ln x_1 \cdot \ln x_2].$$

The last term in this product is a little surprising, but it is easily checked that the criterion of formal justice is satisfied.

For general methods to deal with functional equations in several variables, we refer to **[6]**. We also mention that in view of the comments after Theorem 1, the continuity condition on m can be significantly relaxed. For example, the assertion remains true if for all i, $m(x_1, \dots, x_i, \dots, x_N)$, viewed as a function of x_i alone (with the other x_j as parameters) is continuous for at least one $x_i \in \Re_+$. For an elegant treatment of other functional equations in several variables and their applications to social sciences, see **[3]**.

Proof of Theorem 2. The proof is by induction over N. For $N = 1$, the result is true by the corollary in Section 3. Suppose that the assertion holds for all $L < N$.

To simplify our argument, let

$$g(t_1, \dots, t_N) = \ln m(e^{t_1}, \dots, e^{t_N}).$$

We have to show that

$$g(t_1, \dots, t_N) = \sum_{M \subset \{1, \dots, N\}} p_M \prod_{i \in M} t_i. \tag{8}$$

By equation (6) there are functions k_N and q_N such that

$$g(t_1, \dots, t_{N-1}, t_N) = k_N(t_1, \dots, t_{N-1}) + q_N(t_1, \dots, t_{N-1}) \cdot t_N. \tag{9}$$

On the other hand, if we consider t_N as a parameter, then, by the inductive assumption, there are functions $r_M(t_N)$ such that

$$g(t_1, \dots, t_N) = \sum_{M \subset \{1, \dots, N-1\}} r_M(t_N) \prod_{i \in M} t_i. \tag{10}$$

By identifying the right-hand sides of (9) and (10) and setting $t_N = 0$, we have

$$k_N(t_1, \dots, t_{N-1}) = \sum_{M \subset \{1, \dots, N-1\}} r_M(0) \prod_{i \in M} t_i,$$

and (9) and (10) imply that

$$q_N(t_1, \dots, t_{N-1}) \cdot t_N = \sum_{M \subset \{1, \dots, N-1\}} (r_M(t_N) - r_M(0)) \prod_{i \in M} t_i. \tag{11}$$

If we now set $t_i = 0$ for all $i = 1, \dots, N-1$, we find from (11), with the usual interpretation of an empty product as identically 1,

$$r_\varnothing(t_N) - r_\varnothing(0) = q_N(0, \dots, 0) \cdot t_N, \tag{12}$$

i.e. $r_\varnothing$ is an affine linear function.

We have to introduce some notation before we can proceed. For $M \subset \{1, \dots, N\}$, and $t \in \mathfrak{R}^N$, let $t|_M \in \mathfrak{R}^N$ be defined by

$$(t|_M)_i = \begin{cases} t_i \text{ if } i \in M \\ 0 \text{ if } i \notin M. \end{cases}$$

Generalizing (12), we next claim that there are real constants K_M and C_M, indexed by subsets of $\{1, \dots, N-1\}$, such that

$$q_N(t|_M) = \sum_{\tilde{M} \subset M} K_{\tilde{M}} \prod_{i \in \tilde{M}} t_i \tag{13}$$

and

$$r_M(t_N) - r_M(0) = C_M \cdot t_N \tag{14}$$

for each $M \subset \{1, \dots, N-1\}$.

Once (14) is established, our proof is complete, because we only need to substitute the affine linear functions r_M into (10) to obtain equation (8).

Equation (13) is trivially true for $M = \varnothing$, and (14) holds for $M = \varnothing$ by (12). For general M, we use induction over the cardinality of M. Suppose that (13) and (14) hold for all M such that $|M| \leqslant l < N-1$. To make the induction step, choose an M and k such that $|M| = l$, $k \notin M$, $k \neq N$ and put $t_i = 0$ for all $i \notin M \cup \{k\}$ in equation (11), producing

$$q_N(t|_{M \cup \{k\}}) \cdot t_N = \sum_{\tilde{M} \subset M \cup \{k\}} (r_{\tilde{M}}(t_N) - r_{\tilde{M}}(0)) \prod_{i \in \tilde{M}} t_i. \tag{15}$$

In analogy with (12), the right-hand side of (15) decomposes into

$$\sum_{\tilde{M} \subset \neq M \cup \{k\}} (r_{\tilde{M}}(t_N) - r_{\tilde{M}}(0)) \prod_{i \in \tilde{M}} t_i + (r_{M \cup \{k\}}(t_N) - r_{M \cup \{k\}}(0)) \prod_{i \in M \cup \{k\}} t_i.$$

The induction hypothesis (14) applies to the first term and yields

$$\sum_{\tilde{M} \subset \neq M \cup \{k\}} C_{\tilde{M}} \cdot t_N \prod_{i \in \tilde{M}} t_i.$$

We insert this into (15) and separate t_N from the other variables, obtaining:

$$t_N \left[q_N(t|_{M \cup \{k\}}) - \sum_{\tilde{M} \subset \neq M \cup \{k\}} C_{\tilde{M}} \prod_{i \in \tilde{M}} t_i \right] = [r_{M \cup \{k\}}(t_N) - r_{M \cup \{k\}}(0)] \prod_{i \in M \cup \{k\}} t_i,$$

which implies that

$$\frac{t_N}{r_{M \cup \{k\}}(t_N) - r_{M \cup \{k\}}(0)} = \frac{\prod_{i \in M \cup \{k\}} t_i}{\left[q_N(t|_{M \cup \{k\}}) - \sum_{\tilde{M} \subset \neq M \cup \{k\}} C_{\tilde{M}} \prod_{i \in \tilde{M}} t_i \right]}, \tag{16}$$

except in the (trivial) case where $r_{M \cup \{k\}}(t_N) = r_{M \cup \{k\}}(0)$ for all t_N.

The left-hand side in (16) depends only on t_N, whereas t_N is not present at all on the right. Therefore, both sides must be equal to a constant, and we get (13) and (14) for $M \cup \{k\}$. The proof is complete.

We conclude this section by reformulating Theorem 2 so that the underlying functional equations are transparent. To this end, if $M \subset \{1, \dots, N\}$, $|M| = N-1$ and

i is the only index *not* in M, let $x_M = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathfrak{R}_+^{N-1}$ and $m_{x_M}(x_i) = m(x_1, \dots, x_i, \dots, x_N)$.

COROLLARY. *Let* m: $\mathfrak{R}_+^N \to \mathfrak{R}_+$ *be such that for every* $i \in \{1, \dots, N\}$, $M = \{1, \dots, N\}/\{i\}$ *and* $x_M \in \mathfrak{R}_+^{N-1}$, *the function* $m_{x_M}(x_i)$ *is continuous for at least one* $x_i \in \mathfrak{R}_+$ *and satisfies a functional equation*

$$m_{x_M}\left(\frac{x}{y}\right) = C(x_M) \cdot \frac{m_{x_M}(x)}{m_{x_M}(y)}.$$

Then m is of the form (7).

5. Conclusions We have discovered that in an investigation where a wage function m: $\mathfrak{R}_+^N \to \mathfrak{R}_+$ (which is observed in reality) satisfies our requirement of formal justice, 2^N parameters have to be identified (one for each subset of $\{1, \dots, N\}$). Because of the exponential growth of 2^N with N, it seems hopeless to test formal justice when numerous compensable properties are considered simultaneously. For $N = 2$ or 3, however, it is a feasible task and well worth a sociological study.

We also suggest that market influences may actually be included in the functions permitted. Higher demand for a certain compensable property must not necessarily violate the first isomorphic principle; it can simply lead to a change in the parameters that determine m.

Acknowledgement. I am indebted to David Gartrell, who first brought the questions addressed here to my attention, and to the referee, whose advice simplified the proof of THEOREM 2 and alerted me to the extensive literature on functional equations. This research was supported by NSERC grant Nr. A-7847.

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The Birthday Problem Revisited

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Introduction The birthday problem asks for the probability that at least two people in a group of k people will have the same birthday. The problem continues to attract interest in the classroom and its variations and generalizations provide context for further theoretical elaboration. References to some of this material are included at the end of this note. The present work was motivated by the need to provide an approximation formula for the solution of the birthday problem in a liberal arts course on the Nature of Mathematics. The main result enables students who have not yet studied calculus to approximate solutions to birthday-type problems.

If each person in a group of k people chooses a number at random from a given set of n numbers, the probability p that there will be at least one repetition is

$$p = 1 - \frac{P(n,k)}{n^k}, \quad \text{where } P(n,k) = \frac{n!}{(n-k)!} \quad \text{and} \quad k \le n. \tag{1}$$

For $n = 365$ the formula yields the solution for the birthday problem. The scientific calculators that most students have can calculate p for some n and k, but their range is quite limited. It is desirable to use approximation formulas that yield p to a reasonable degree of accuracy. One such formula is $p > 1 - e^{-k(k-1)/2n}$ [4, p. 33]. In this note we will derive another approximation that improves on this and is more elementary. The derivation is based on the relationship between the geometric and the arithmetic means of a set of positive numbers [7, p. 18]. We will also use Taylor's approximations to derive a formula for the upper bound of the error. That will be accessible to students with a knowledge of elementary calculus.

An elementary approximation formula for p We show that

$$p > 1 - \left(1 - \frac{k}{2n}\right)^{k-1}, \quad \text{for } k \le n. \tag{2}$$

The derivation involves first finding an upper bound for $q = P(n,k)/n^k$ from which the above lower bound for $p = 1 - q$ will follow. If the numerator and the denominator of q are written out and common terms are cancelled we obtain

$$q = \frac{P(n,k)}{n^k} = \frac{P(n-1,k-1)}{n^{k-1}} = \prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right). \tag{3}$$

Since the numbers in the product are not identical, the geometric mean is strictly less than the arithmetic mean. Taking the respective means yields the inequality

$$\left[\prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right)\right]^{\frac{1}{k-1}} < \frac{\sum_{j=1}^{k-1}\left(1 - \frac{j}{n}\right)}{k-1} = \frac{k-1-\sum_{j=1}^{k-1}\frac{j}{n}}{k-1} = \left(1 - \frac{k}{2n}\right). \tag{4}$$

The desired upper bound for q is given by

$$q = \prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right) < \left(1 - \frac{k}{2n}\right)^{k-1}. \tag{5}$$

For $k = 23$ in the birthday problem the exact value of p correct to four decimal places is $p = 0.5073$; the approximation formula yields $p > 0.5055$. As another example we may suppose that in a group of 55 people each person writes down a three-digit number independently from others. Here, for $k = 55$ and $n = 900$ the probability p of at least two people writing the same number is $p = 0.8144$, correct to four decimal places. The approximation formula gives a lower bound as $p > 0.8128$. The relatively small size of the error in these approximations is due to the near equality of the numbers $(1 - (j/n))$ involved in going from the geometric mean to the arithmetic mean.

To compare the usual approximation with this one, we define p_1, q_1, p_2, and q_2 by

$$p_1 = 1 - q_1 = 1 - \left(1 - \frac{k}{2n}\right)^{k-1}$$

and (6)

$$p_2 = 1 - q_2 = 1 - e^{-k(k-1)/2n}.$$

Then

$$\ln q_1 = (k-1)\ln\left(1 - \frac{k}{2n}\right) < -(k-1)\frac{k}{2n}.$$

The last inequality follows from $\ln(1-x) < -x$ for $|x| < 1$ (look at the Taylor expansion of $\ln(1-x)$ about the origin). And clearly,

$$q_1 < e^{-k(k-1)/2n} = q_2.$$

Thus $q < q_1 < q_2$ and

$$p > p_1 > p_2. \tag{7}$$

In the approximation of p, p_1 is an improvement over p_2.

The error in the approximation The error E in the approximation of p by p_1 satisfies the following inequalities

$$E = p - p_1 < p - p_2 < q_2(1 - e^{-\varepsilon}) < q_2\varepsilon, \qquad k \le n. \tag{8}$$

We will show that

$$\varepsilon < \frac{k^3}{6(n-k+1)^2} \tag{9}$$

by using a straightforward application of Taylor's linear approximation with an error term for $\ln(1-x)$ around the origin, when $0 < x < 1$. That is, $\ln(1-x) = -x - (x^2/2(1-\xi)^2)$ where $0 < \xi < x$. Applying this to $\ln q$ in (3) we have

$$\ln q = \ln \prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right) = \sum_{j=1}^{k-1} \ln\left(1 - \frac{j}{n}\right) = \sum_{j=1}^{k-1} -\frac{j}{n} - \frac{\left(\frac{j}{n}\right)^2}{2(1-\xi_j)^2},$$

where $0<\xi_j<j/n\leq(k-1/n)$. The first terms in the second sum add up to $-(k(k-1)/2n)$ and if we denote the rest by

$$\varepsilon=\sum_{j=1}^{k-1}\frac{\left(\frac{j}{n}\right)^2}{2(1-\xi_j)^2}, \tag{10}$$

we obtain $\ln q=-(k(k-1)/2n)-\varepsilon$ and

$$q=e^{-\frac{k(k-1)}{2n}}e^{-\varepsilon}=q_2e^{-\varepsilon}. \tag{11}$$

The last inequality in (8) may be verified by looking at the error term in the linear Taylor's approximation of $f(x)=1-e^{-x}$ about the origin. This concludes the proof of (8).

To get the upper bound (9) for ε we use the condition $0<\xi_j<(j/n)\leq(k-1/n)$ to obtain

$$\frac{1}{(1-\xi_j)^2}<\frac{n^2}{(n-k+1)^2}$$

and apply it to ε in (10) to yield

$$\varepsilon<\sum_{j=1}^{k-1}\frac{j^2}{2(n-k+1)^2}=\frac{1}{2(n-k+1)^2}\left(\frac{k(k-1)(2k-1)}{6}\right).$$

Hence,

$$\varepsilon<\frac{k^3}{6(n-k+1)^2}.$$

For $n=10^{10}$, $k=10^5$ we have $p_1=0.3934670$, $p_2=0.3934663$, and $E<1.01\times10^{-6}$. In this case the approximation formulas give p correct to five decimal places. In general, as we shall see below, the error in the approximations p_1 and p_2 tends to zero as n and k go to infinity.

The two terms on the right-hand side of the error inequality $E<q_2(1-e^{-\varepsilon})$ in (8) are nonnegative and are bounded above by 1. If n is very large compared to k, then q_2 tends to zero, otherwise $(1-e^{-\varepsilon})$ diminishes for large k. More precisely, if $n\leq k^{1.75}$, then

$$q_2=e^{-(k(k-1)/2n)}\leq e^{-(k(k-1)/2k^{1.75})}.$$

For arbitrary $\varepsilon>0$ we may choose $K_1>0$ so large that $e^{-(k(k-1)/2k^{1.75})}<\varepsilon$ for $k>K_1$. Then, $E<q_2(1-e^{-\varepsilon})<\varepsilon$.

On the other hand, for $n>k^{1.75}$, we have the inequalities

$$1-e^{-\varepsilon}<1-e^{-(k^3/6(n-k+1)^2)}<1-e^{-(k^3/6(k^{1.75}-k+1)^2}.$$

The last difference tends to zero for large values of k. $K_2>0$ can now be chosen so large that for $k>K_2$ the right-hand term is less than ε. Letting $K=\max\{K_1,K_2\}$ and $N=K$ will assure that $E<\varepsilon$ for $k>K$ and $n>N$. Thus the error E goes to zero as n and k go to infinity.

It is interesting to note that replacing $E<q_2(1-e^{-\varepsilon})$ by $E<q_2\varepsilon$ is not a huge loss even when ε is large. In that case q_2 will be "exponentially" small and will dominate the product.

A limiting behavior of p If k and n are related by $k = cn^{\alpha}$, where $c > 0$ and $\alpha > 0$, then p has an interesting behavior in the limit, as displayed below:

$$\lim_{n\to\infty} p = \begin{cases} 0 & \text{if } \alpha < \frac{1}{2} \\ 1 - e^{-\frac{c^2}{2}} & \text{if } \alpha = \frac{1}{2} \\ 1 & \text{if } \alpha > \frac{1}{2} \end{cases} \cdot \tag{12}$$

Since the error in the approximation of p by p_2 is zero in the limit, it is sufficient to verify the above results for p_2. Let

$$k = cn^{\alpha} = cn^{\frac{1}{2}+\beta}$$

where the three cases are determined by taking β as positive, negative or zero. Making the substituting $k^2 = c^2 n n^{2\beta}$ in q_2 we have

$$\lim_{n\to\infty} q_2 = \lim_{n\to\infty} e^{-(k(k-1)/2n)} = \lim_{n\to\infty} e^{-(k^2/2n)} = \lim_{n\to\infty} e^{-(c^2 n^{2\beta}/2)}.$$

The conclusions in (12) readily follow from

$$\lim_{n\to\infty} p = \lim_{n\to\infty} (1 - q_2)$$

by examining three different cases of β.

The interesting case of

$$\lim_{n\to\infty} p = e^{-(c^2/2)} \quad \text{for } k = cn^{1/2}$$

enables us to estimate k for fixed n that will make the probability p a preassigned value. For example, to obtain $p > \frac{1}{2}$ we get an estimate for k as $k > \sqrt{2\ln 2}\sqrt{n}$. In the birthday problem with $n = 365$ this formula gives $k > 22.49$. This is actually correct when k is taken as an integer. For $k = 23$, the exact probability is $p = 0.507$.

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PROBLEMS

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Texas Christian University

Proposals

To be considered for publication, solutions should be received by November 1, 1994.

1448. *Proposed by Daniel P. Moore, Alexander Consulting Group, Chicago, Illinois.*

Let $q_1, q_2, \ldots, q_n$ be odd integers, $n \geq 2$. Prove or disprove the following. There are integers $d_1, d_2, \ldots, d_n$, each equal to 0, 1, or -1, but not all zero, such that

$$\sum_{i=1}^{n} d_i q_i \equiv 0 \pmod{2^n}.$$

1449. *Proposed by Paul Bracken, University of Waterloo, Waterloo, Ontario, Canada.*

Let $a_1 = 1$, and $a_n = n(a_{n-1} + 1)$ for $n > 1$. Compute the product

$$\prod_{n=1}^{\infty} \left(1 + a_n^{-1}\right).$$

1450. *Proposed by John O. Kiltinen, Northern Michigan University, Marquette, Michigan.*

Which permutations in S_n, the group of all permutations on the set $\{1, 2, \ldots, n\}$, can be expressed as a product of two n-cycles?

1451. *Proposed by Barry Cipra, Northfield, Minnesota.*

Let f be an entire function such that $f(z) + f(z+1) = f(2z)$ and $f(0) = 0$. Prove that f is identically zero.

 We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, 1520 St. Olaf Ave., Northfield, MN 55057-1098 or mailed electronically via fax: (507) 663-3549 or e-mail: larson@stolaf.edu.

1452. *Proposed by John Frohliger and Adam Zeuske (student), St. Norbert College, DePere, Wisconsin.*

Let ABC be a given triangle and θ an angle between $-90°$ and $90°$. Let A',B',C' be points on the perpendicular bisectors of BC, CA, and AB respectively, so that $\angle BCA'$, $\angle CAB'$, and $\angle ABC'$ all have measure θ. Show that for all but two values of θ, the lines AA', BB', and CC' are concurrent, provided that points A', B', C' are not equal to A,B,C, respectively.

1453*. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Let $E_{n+1} = x^{E_n}$ where $E_0 = 1, n = 0, 1, \ldots,$ and $x \geq 0 (E_1 = x, E_2 = x^x$, etc.). It is easy to show that for $x \geq 1$, $E_n (n > 1)$ is a strictly increasing convex function. Prove or disprove each of the following.

(i) E_{2n} is a unimodal convex function for $n > 1$ and all $x \geq 0$.

(ii) E_{2n+1} is a increasing function for $x \geq 0$, and is concave in a small enough interval $[0, \varepsilon(n)]$.

Quickies

Answers to the Quickies are on page 229.

Q820. *Proposed by Cristian Turcu, London, England.*

Let n be an odd positive integer and $u_1, u_2, \ldots, u_n$ be complex numbers whose product is 1. Assuming that

$$u_1 + \frac{1}{u_1} = u_2 + \frac{1}{u_2} = \cdots = u_n + \frac{1}{u_n},$$

show that $u_1, u_2, \ldots, u_n$ are each m-th roots of unity for some odd integer $m \leq n$.

Q821. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.*

Let p be a prime. Show that for any nonnegative integers n and k, with $0 \leq k \leq p^n - 1$,

$$\binom{p^n - 1}{k} \equiv (-1)^k \pmod{p}.$$

Q822. *Proposed by Ismor Fischer, University of Wisconsin, Madison, Wisconsin.*

Find the value of $\lim_{x \to 0} \left(\dfrac{\sin x}{x} \right)^{1/x^2}$.

Solutions

Two Triangles Within a Triangle **June 1993**

1423. *Proposed by Mirel Mocanu, University of Craiova, Craiova, Romania.*

Two equilateral triangles, of side-lengths a and b respectively, are enclosed in a unit equilateral triangle so that they have no common interior points. Prove that $a + b \leq 1$.

Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

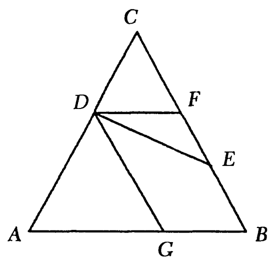
Let A,B,C be the vertices of the enclosing unit triangle. Since the two enclosed triangles have no common interior points, there is a line that separates their interiors. Note that this line must intersect the triangle ABC in exactly two points, say D and E, one of which, say D, is not a vertex point. We may assume that D lies on the side AC and E on the side BC such that $|DC| \leqslant |CE|$. Suppose the line passing through D and parallel to AB intersects the triangle ABC at F, and the line passing through D and parallel to CB intersects the triangle ABC at G. One of the enclosed triangles, call it T, of side length a, lies between the parallel lines DF and AB. The width of T in the direction of a given line L is defined by

$$w(L) = \sup\{|PQ| : P \text{ and } Q \text{ lie in } T \text{ and } PQ \text{ is parallel to } L\}.$$

If d is the distance between DF and AB, then there must be a line L_0 such that $w(L_0) \le d$. The line L_1 containing any median of T yields the minimum value of w. Therefore

$$\frac{\sqrt{3}}{2}a = w(L_1) \le w(L_0) \le d = \frac{\sqrt{3}}{2}|AD|.$$

It follows that $a \le |AD|$.



The other enclosed triangle lies between the lines DG and CB. By an argument similar to the one above, we may show that $b \le |DC|$. Since $|AD| + |DC| = 1$, we have shown that $a + b \le 1$.

Also solved by Eric Chandler, Jiro Fukuta (Japan), Michael Golomb, Kiran Kedlaya (student), O. P. Lossers (The Netherlands), Andreas Müller (Germany), László Szücs, A. N. 't Woord (The Netherlands), and the proposer. There were three incomplete solutions.

Complete Set of Residues Modulo n **June 1993**

1424. *Proposed by J. C. Binz, University of Bern, Bern, Switzerland.*

Find all positive integers n such that

$$M_n = \left\{\binom{1}{2},\binom{2}{2},\binom{3}{2},\dots,\binom{n}{2}\right\}$$

is a complete set of residues modulo n.

Solution by David Hankin, John Dewey High School, Brooklyn, New York.

The elements of M_n will be a complete set of residues if, and only if, $\binom{x}{2} \equiv \binom{y}{2}$ (mod n), $1 \le x < y \le n$ has no solutions. We will show this is the case if, and only if, n is a power of 2.

First, note that $\binom{x}{2} \equiv \binom{y}{2}$ (mod n) if, and only if, $(y-x)(y+x-1) \equiv 0 \pmod{2n}$.

Suppose n is a power of 2; then so is $2n$. Since $y - x$ and $y + x - 1$ are of different parity and both are less than $2n$, it is clear that $2n$ cannot divide their product. Thus, M_n must be a complete set of residues modulo n.

Suppose n is not a power of 2, and write $n = 2^e m$, where m is odd, $m \geq 3$ and $e \geqslant 0$. Set $y - x = \min\{2^{e+1}, m\}$ and $y + x - 1 = \max\{2^{e+1}, m\}$. Then $y = (2^{e+1} + m + 1)/2$ and $x = (|2^{e+1} - m| + 1)/2$. Then, $1 \leq x < y \leq n$ and from above, $\binom{x}{2} \equiv \binom{y}{2}$ (mod n).

Also solved by Brian D. Beasley, William D. Blair, Stephen D. Bronn, David Callan, Duff Campbell, Con Amore Problem Group (Denmark), Bill Correll, Jr. (student), Luis V. Dieulefait (student, Argentina), Robert L. Doucette, Milton P. Eisner, F. J. Flanigan, Arthur H. Foss, Michael Golomb, Russell Jay Hendel, Mack Hill, Richard Holzsager, Thomas Jager, Javelina Problem Solvers, Kiran Kedlaya (student), Kee-Wai Lau (Hong Kong), Detlef Laugwitz (Germany), James T. Lewis, Peter W. Lindstrom, Nick Lord (England), O. P. Lossers (The Netherlands), Helen M. Marston, Reiner Martin (student), MATC Problem Solving Group, Robert Patenaude, F. C. Rembis, John Rickert, R. P. Sealy (Canada), Heinz-Jürgen Seiffert (Germany), Man-Keung Siu (Hong Kong), John S. Sumner, Michael Vowe (Switzerland), Edward T. H. Wang (Canada), A. N. 't Woord, and the proposer.

L. R. King, Davidson College, points out a link between this problem and the known fact that any integer that is not a power of two can be written as the sum of (two or more) consecutive integers. For more on this connection, see the comments to Problem 1358, MATHEMATICS MAGAZINE, **64**, No. 5, p. 351.

A Family of Nonsingular Matrices — June 1993

1425. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.*

Let p be a prime number and let $\mathbf{A}$ be a $(p-1) \times (p-1)$ matrix over the field of rational numbers such that $\mathbf{A}^p = \mathbf{I} \neq \mathbf{A}$. Show that if $f(x)$ is any nonzero polynomial with rational coefficients and degree less than $p - 1$, then $f(\mathbf{A})$ is nonsingular.

Solution by William D. Blair, Northern Illinois University, Dekalb, Illinois.

Let $\phi(x) = x^{p-1} + x^{p-2} + \cdots + 1$. It is well known that $\phi(x)$ is irreducible over the rational numbers (corollary to Eisenstein's Criterion). Since $\mathbf{A}$ satisfies $x^p - 1$, the minimal polynomial of $\mathbf{A}$ divides $x^p - 1$. Since the minimal polynomial of $\mathbf{A}$ has degree at most $p - 1$, since $\mathbf{A} \neq \mathbf{I}$, and since the factorization $x^p - 1 = (x-1)\phi(x)$ into irreducible polynomials is unique, we conclude that $\phi(x)$ is the minimal polynomial of $\mathbf{A}$. If $f(x)$ is any nonzero polynomial with rational coefficients and degree less than $p - 1$, then $f(x)$ and $\phi(x)$ are relatively prime, and there exist polynomials $g(x)$ and $h(x)$ with rational coefficients such that $f(x)g(x) + \phi(x)h(x) = 1$. Since $\phi(\mathbf{A}) = \mathbf{0}$, we have $f(\mathbf{A})g(\mathbf{A}) = \mathbf{I}$ and $f(\mathbf{A})$ is nonsingular.

Also solved by Sinefakopoulos Achilleas (student, Greece), David Callan, Ron Martin Carroll, Con Amore Problem Group (Denmark), Luz M. DeAlba, Robert L. Doucette, F. J. Flanigan, Richard Holzsager, Thomas Jager, Kiran Kedlaya (student), Hubert Kiechle (Germany), John F. Kurtzke, Nick Lord (England), O. P. Lossers (The Netherlands), MATC Problem Solving Group, F. C. Rembis, Rockford College Problem Group, A. N.'t Woord (The Netherlands), and the proposer.

A Pinwheel of Chords — June 1993

1426. *Proposed by Jiro Fukuta, Gifu-ken, Japan.*

Consider a circle with center at O, and a regular n-gon, $A_1A_2 \ldots A_n$, contained entirely within the given circle. Let C denote the center of the n-gon. Let P_iQ_i, $i = 1, 2, \ldots, n$ be the chords of the given circle that are perpendicular to CA_i at A_i. Prove that $\sum_{i=1}^{n} (CP_i^2 + CQ_i^2)$ is a constant.

Solution by A. N. 't Woord, Eindhoven University of Technology, Eindhoven, The Netherlands.

For the circle we take the unit circle in the complex plane. After rotation we may assume that the n-gon is of the form $A_j = C + r\omega^j$ with $C \in \mathbf{C}$, $\omega = e^{2\pi i/n}$ and $r \in (0, 1)$.

A parametrization of P_jQ_j is given by $A_j + \lambda i \omega^j$ $(\lambda \in \mathbf{R})$, so we can write $P_j = A_j + \lambda_p i \omega^j$ and $Q_j = A_j + \lambda_q i \omega^j$. Now we can calculate

$$CP_j^2 + CQ_j^2 = |P_j - C|^2 + |Q_j - C|^2 = |\omega^j(r + \lambda_p i)|^2 + |\omega^j(r + \lambda_q i)|^2 = 2r^2 + \lambda_p^2 + \lambda_q^2.$$

Note that if $(X-u)(X-v) = X^2 + bX + c$ then $u^2 + v^2 = b^2 - 2c$. Since λ_p and λ_q are the two distinct roots of

$$0 = |A_j + \lambda i \omega^j|^2 - 1 = \lambda^2 + \lambda i\left(\overline{A}_j\omega^j - A_j\omega^{-j}\right) + A_j\overline{A}_j - 1,$$

we find

$$\begin{aligned}\lambda_p^2 + \lambda_q^2 &= -\left(\overline{A}_j\omega^j - A_j\omega^{-j}\right)^2 - 2\left(A_j\overline{A}_j - 1\right)\\ &= 2 - \left(\left(\overline{A}_j\omega^j\right)^2 + \left(A_j\omega^{-j}\right)^2\right)\\ &= 2 - \left(\left(\overline{C}\omega^j + r\right)^2 + \left(C\omega^{-j} + r\right)^2\right)\\ &= 2 - 2r^2 - \left(\overline{C}\omega^{2j} + C\omega^{-2j} + 2r\left(\overline{C}\omega^j + C\omega^{-j}\right)\right).\end{aligned}$$

Using these results we have

$$\sum_{j=1}^{n}\left(CP_j^2 + CQ_j^2\right) = \sum_{j=1}^{n}\left(2 - 2r\left(\overline{C}\omega^j + C\omega^{-j}\right) - \overline{C}\omega^{2j} + C\omega^{-2j}\right) = 2n,$$

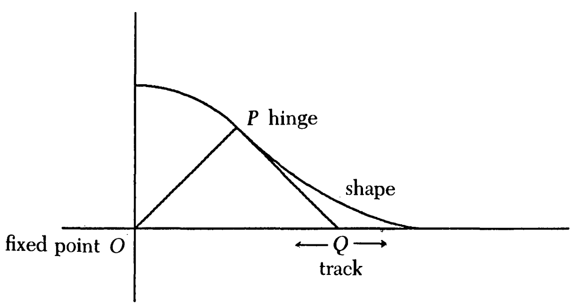
because $\sum_{j=1}^{n}\omega^j = \sum_{j=1}^{n}\omega^{2j} = 0$.

Also solved by J. C. Binz (Switzerland), Michael Golomb, Richard Holzsager, Kiran Kedlaya (student), Nick Lord (England), O. P. Lossers (The Netherlands), F. C. Rembis, James S. Robertson, and the proposer.

A Bi-fold Closet Door **June 1993**

1427. *Proposed by Gary L. Van Velsir, Anne Arundel Community College, Arnold, Maryland.*

A bi-fold closet door consists of two one-foot-wide panels, hinged at point P. One of the panels is fixed at the point O. Assume that as the endpoint Q moves to the right, the door rubs against a thick carpet. What shape will be swept out on the carpet?



Solution by the New Mexico Tech Problem Solving Group, New Mexico Institute of Mining and Technology, Socorro, New Mexico.

If point O is at the origin, and point Q slides along the positive x axis, the shape will be the region in the first quadrant bounded by the arcs $x^2 + y^2 = 1$ from $0 \le x \le 1/\sqrt{2}$, and $x^{2/3} + y^{2/3} = 2^{2/3}$ from $1/\sqrt{2} \le x \le 2$, where all distances are measured in feet.

The origin of the circular arc is clear, since point P follows a circular path while the door is closing. Point P continues to determine the shape when the panel PQ is tangent to the circle. This happens when P is at $(1/\sqrt{2}, 1/\sqrt{2})$, since then angle OPQ is 90°. To find the shape beyond there, let us denote the angle of elevation to P (angle QOP) by θ, so that θ decreases from $\pi/2$ to 0 as the door closes. For a given x, denote the corresponding y coordinate on the isosceles triangle formed by the panels OP and PQ by $h(x,\theta)$. Since OP and PQ are of length one, we see that $h(0,\theta) = h(2\cos\theta,\theta) = 0$, $h(\cos\theta,\theta) = \sin\theta$, and in general,

$$h(a\cos\theta,\theta) = h((2-a)\cos\theta,\theta) = a\sin\theta,$$

where $0 \le a \le 1$. Since we are interested in the shape to the right of point P, we eliminate a using the second value of x: $x = (2-a)\cos\theta$, or $a = 2 - x\sec\theta$. Thus, if $\cos\theta \le x \le 2\cos\theta$, we have $h(x,\theta) = 2\sin\theta - x\tan\theta$.

We would like to find the largest value of h for a *fixed* value of x, so we solve

$$\frac{\partial h}{\partial\theta} = 0.$$

This occurs when $2\cos\theta = x\sec^2\theta$, or $x = 2\cos^3\theta$. To find the critical y value, we substitute this back into h, giving

$$h = 2\sin\theta(1-\cos^2\theta) = 2\sin^3\theta$$

as the highest point for a given value of x.

Thus we have $(x,y) = (2\cos^3\theta, 2\sin^3\theta)$, or equivalently, $x^{2/3} + y^{2/3} = 2^{2/3}$ for the arc from $1/\sqrt{2} \le x \le 2$.

Also solved by Henry J. Barten, S. J. Becker, Daniel Braithwaite (student), Eric Chandler, Con Amore Problem Group (Denmark), David Doster, Robert L. Doucette, Milton P. Eisner, Ervin Eltze, Arthur H. Foss, Robert Geretshläger (Austria), N. N. Gurwell and E. D. Onstott, David Hankin, Francis M. Henderson, Richard Holzsager, Javelina Problem Solvers, Jon L. Johnson, The Northern Kentucky University Problem Group, Peter A. Lindstrom, Nick Lord (England), O. P. Lossers (The Netherlands), MATC Problem Solving Group (two solutions), Andreas Müller (Germany), Hugh Noland, Stephen Noltie, Jeremy Ottenstein (Israel), Cornel G. Ormsby, F. C. Rembis, Larry Riddle, Man-Keung Siu (Hong Kong), Nora S. Thornber, Barbara Victor, Jack V. Wales, Jr., Michael S. Waters (student), Harry Weingarten, Matthew F. Wyneken, Harold Ziehms, and the proposer. There was one unsigned solution.

Coincidentally, this problem is the subject of an article by Jack Weiner and G. R. Chapman, "Inflections on the Bedroom Floor," *Mathematics Teacher*, Vol 86, No. 7, October 1993, pp. 598–601. A related problem was posed by Man-Keung Siu in the *Hong Kong Mathematical Society Newsletter*, No. 3, June 1987.

Answers

Solutions to the Quickies on page 224.

A820. Let $\alpha = u_1 + 1/u_1 = \cdots = u_n + 1/u_n$. Then for each i, $u_i \in \{\varepsilon, 1/\varepsilon\}$, where ε and $1/\varepsilon$ are the two roots of $z^2 - \alpha z + 1 = 0$. Because the product of the u_i's is 1 and n is odd, it must be the case that $\varepsilon^m = 1$ or $(1/\varepsilon)^m = 1$ for some odd integer $m \le n$. This completes the proof. (Note: The result is not true when n is even; for example, take $n = 4$, $\alpha = \sqrt{5}$, $u_1 = u_2 = (\sqrt{5}+1)/2$, $u_3 = u_4 = (\sqrt{5}-1)/2$.)

A821. In the ring $\mathbf{Z}_p[x]$ of polynomials over the p element field $\mathbf{Z}_p$,

$$\sum_{k=0}^{p^n-1}\binom{p^n-1}{k}(-1)^k x^k=(1-x)^{p^n}=\frac{(1-x)^{p^n}}{1-x}=\frac{1-x^{p^n}}{1-x}=\sum_{k=0}^{p^n-1}x^k.$$

Equating coefficients of x^k gives the result.

A822. From the Maclaurin expansion for $\sin x$, we find that

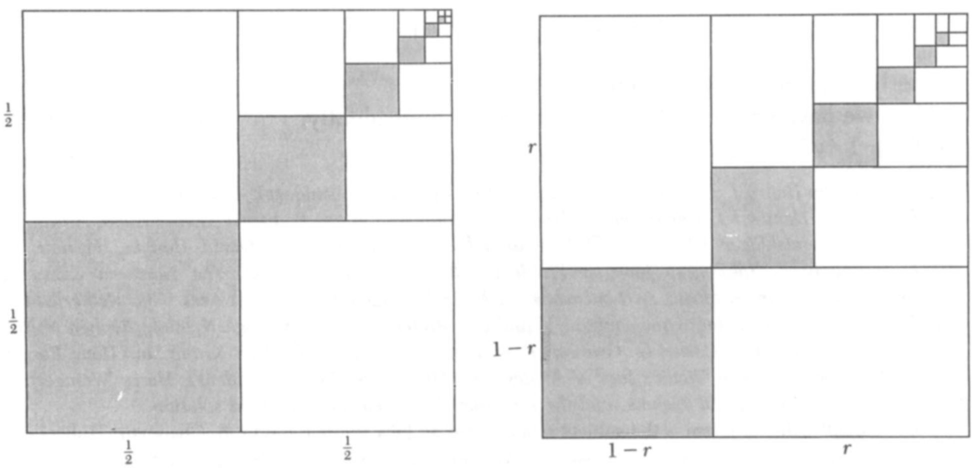
$$\sin x=\left(x+\frac{x^3}{6}\right)\left[1+O(x^4)\right].$$

Hence

$$\left(\frac{\sin x}{x}\right)^{1/x^2}=\left(1-\frac{x^2}{6}\right)^{1/x^2}\left(\left[1+O(x^4)\right]^{1/x^4}\right)^{x^2},$$

from which it follows that the desired limit is $e^{-1/6}$.

Proof without Words: Geometric Series



$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}$$

$$(1-r)^2+r^2(1-r)^2+r^4(1-r)^2+\cdots=\frac{(1-r)^2}{(1-r)^2+2r(1-r)}=\frac{1-r}{1+r}$$

$$1+r^2+r^4+\cdots=\frac{1}{1-r^2}$$

$$a+ar+ar^2+\cdots=\frac{a}{1-r}$$

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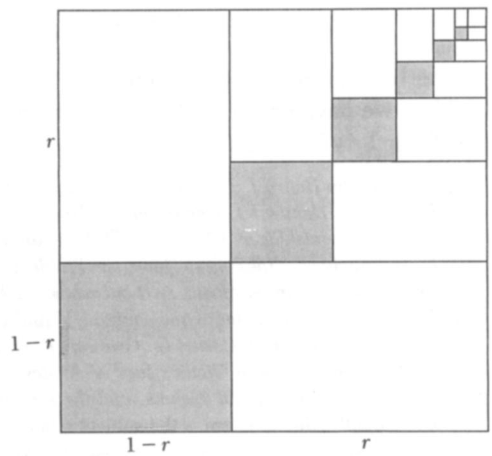

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$$1+r^2+r^4+\cdots=\frac{1}{1-r^2}$$

$$a+ar+ar^2+\cdots=\frac{a}{1-r}$$

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REVIEWS

PAUL J. CAMPBELL
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Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Brams, Steven J., and Samuel Merrill III, Would Ross Perot have won the 1992 presidential election under approval voting?, *PS: Political Science and Politics* (March 1994) 34–44.

Ross Perot has had broad appeal to Republicans, Democrats, and independents. Republicans like his fiscal policies, particularly his emphasis on a balanced budget; Democrats favor his social views; and independents just love his independence (he's like them that way). Wide-ranging appeal is particularly advantageous under approval voting, under which voters can vote for as many candidates as they find acceptable. This article attempts to discover what might have happened in the 1992 election if approval voting had been used. Could Perot have been enough people's second-place choice to have won? The authors use data collected by the Center for Political Studies to suggest a conclusion—but I won't give away the ending here.

Kolata, Gina, The assault on 114,381,625,757,888,867,669,235,779,976,146,612,010,218,296, 721,242,362,562,561,842,935,706,935,245,733,897,830,597,123,563,958,705,058,989,075,147, 599,290,026,879,543,541, *New York Times* (National Edition) (22 March 1994) B5, B9.

The number in the title of this article is better known to its friends as RSA-129, after the RSA cryptosystem and the number's 129 digits. The inventors of RSA proposed in 1977 that it would take others at least 40 quadrillion years to discover the factors of RSA-129. Over the past eight months, however, hundreds of volunteered computers have been working on millions of subproblems of this problem and reporting their results over the Internet back to a central site. By the time you read this, the factors should have been announced. The attack has been based on the quadratic sieve, and a number only slightly bigger would be immensely more difficult to factor. However, what was utterly inconceivable only 17 years ago has been (almost) done.

Andrew Wiles, *People* (27 December 1993—3 January 1994) 104. Jackson, Allyn, Update on proof of Fermat's last theorem: Gap appears in proof but experts laud Wiles's accomplishment, *Notices of the American Mathematical Society* 41 (3) (March 1994) 185–186. Cook, Roger, Fermat's last theorem—a theorem at last, *Mathematical Spectrum* 26 (3) (1993/4) 65–73. Ribet, Kenneth A., and Brian Hayes, Fermat's last theorem and modern arithmetic, *American Scientist* 82 (April 1994) 144–156.

Is Andrew Wiles the first mathematician to appear in *People*? Jackson expands on the "stumbling block" to concluding the proof of FLT, and Cook gives a brief exposition at a level suitable for mathematics majors. The real prize in this group is the splendid exposition for a popular audience, by Ribet and Hayes, of the context and particulars of Wiles's approach to Fermat's last theorem. With this outline as a guide, some suitably chosen elliptic curves as examples, and an enthusiastic teacher, students—even in a course of "math for liberal arts students"—could enjoy exploring this exciting contemporary research area.

Peterson, Ivars, Who's really #1? Choose your math and get the rankings you want, *Science News* 144 (18 & 25 December 1993) 412–413. Watson, Andrew, When is a badminton player like a prize bull?, *New Scientist* (18 December 1993) 4–5. Keener, James P., The Perron-Frobenius theorem and the ranking of football teams, *SIAM Review* 35 (1993) 80–93.

By the time you read this, the college football season of 1993, with its argument about which team should be ranked first, will be over; even basketball's March madness will be gone. Outlasting even the professional hockey season, however, is the question: What procedure gives a just ranking of teams? Where are mathematicians when the public needs them, with an indisputable solution to this problem? Well ... wouldn't that take away some of the fun of being a sports fan? Although using a mathematical ranking scheme takes away some subjectivity, there is no agreement on which to use. Neither is there yet a theorem, like Arrow's impossibility theorem about voting, to say that an ideal ranking system is beyond our reach. In Australia, apparently, ranking bulls—by how much milk their daughters produce—is easier; and a system similar in spirit will be used to rank badminton players for the Olympics in Sydney in 2000.

Hayes, Brian, Balanced on a pencil point, *American Scientist* 81 (November-December 1993) 510–516. Meadows, Donella H., Dennis L. Meadows, and Jørgen Randers. *Beyond the Limits: Confronting Global Collapse, Envisioning a Sustainable Future*, Chelsea Green Publ. Co., 1992.

The largest impact that mathematical modeling has made on public consciousness may have been in 1972, when the Club of Rome published the conclusions of its computer modeling efforts in *The Limits to Growth.* The message was that the world was in danger from overpopulation and pollution; all scenarios led to disaster, sooner or later. Now, more than 20 years later, members of the same group have a new book; but don't look for any change in the conclusions. This time, however, you too can have "doomsday on a desktop," as the computers models themselves are available for $30, in a Stella II version for Macintosh and in a Dynamo Plus version for IBM PC (see the Hayes article for details). "According to the model, we are at a singular moment in the history of the world—the one and only transition from abundance to scarcity, from growth to stasis."

Brams, Steven J., Game theory and literature, *Games and Economic Behavior* 6 (1994) 32–54.

Brams surveys published articles that have applied game theory to works of literature, ranging from the Bible through Shakespeare's plays to Poe's short stories and modern novels. That game theory can be applied to literature, but has not been applied to art or music (as Brams remarks), may be attributable to game theory's concern with discerning what constitutes rational behavior, vis-á-vis the concern of literary authors about the motivation and basis of behavior for their characters. Does Richard III in Shakespeare's play make bad decisions, or is he "eminently rational" (as Brams believes)? Is a tragic fall made more tragic when it proceeds inevitably through rational choices? Even authors who are interested in developing characters who explore deviations from rationality may find studying game theory worthwhile, as a way of discerning what can be rational. Game theory can clarify the choices and strategies available to characters, including the uncertainty inherent in mixed strategies. Brams concludes by suggesting that what can make literature "compelling" rather than just a "humdrum illustration" of a game is the further game that goes on between author and reader. That game consists in the author revealing, and the reader coming to know, more information; the "payoffs" to these two players depend on whether the reader's expectations are realized or disappointed, or even manipulated.

Torture test nets a new prime, *Science* 263 (7 January 1994) 27.

David Slowinski and Paul Gage of Cray Research have used a CRAY Y-MP M90-series supercomputer to find a new largest-known Mersenne prime, $2^{859,433}-1$, which has 258,716 decimal digits. It is the thirty-third Mersenne number that is known to be prime. Other smaller ones may yet lie in the exponent gaps 386,000–430,000 and 524,000–750,000.

Gray, Jeremy, Parsimonious polyhedra, *Nature* 367 (17 February 1994) 598–599.

"What space-filling arrangement of cells of equal volume has minimum surface area?" Lord Kelvin proposed the current record-holder: the 14-sided truncated octahedron, with its sides curved and faces bowed so that is a little more like a sphere. One hundred years later, D. Weaire and R. Phelan have discovered another 14-sided cell that improves on Lord Kelvin's by 0.3%. But is it optimal?

Ekert, Artur, Shannon's theorem revisited, *Nature* 367 (10 February 1994) 513–514.

Progress in optics has resulted in the construction of quantum cryptosystems, in which Ekert was involved. He suggests that "quantum data processing" and "quantum computation" may be just around the corner—quantum effects being used for "coding, signalling, transmission or detection." Shannon's theorem for the non-quantum world says that optimal data compression is achieved by a binary code for which the mean number of bits per symbol is at least as large as the Shannon entropy of the source, defined as $\sum -p_i \lg p_i$, where p_i is the frequency of the ith symbol transmitted. (For ordinary English text, this optimum is about 2.8 bits per letter.) Shannon's theorem assumes that "we can always reliably distinguish between different symbols"—a premise invalid in the quantum world. In that world, quantum states are vectors in a Hilbert space, but only orthogonal vectors can be perfectly distinguished from each other. The classical Shannon theorem applies if all of the quantum states in the transmission are mutually orthogonal. In the more general case, though, what is the minimum number of physical bits ("qubits") required for coding? Benjamin Schumacher has shown that the lower bound, known as *von Neumann entropy*, is never greater than the Shannon entropy, equaling it for the case of mutual orthogonality. Says Ekert: "This generalization of the Shannon theorem comes just in time."

Gribbin, John, The prescient power of mathematics, *New Scientist* (22 January 1994) 14.

Many mathematicians, philosophers, and physicists have contemplated the "unreasonable effectiveness" of mathematics in describing physical phenomena. Bruno Augenstein (Rand Research Institute) claims that the reason may be that physicists can find a "counterpart" to any mathematical concept and urges that physicists "deliberately and routinely" pursue "physical models of already discovered mathematical structures." That may work well for something like non-Euclidean geometry, developed by Riemann and applied in Einstein's relativity theory. But what about something more esoteric and less intuitive? What could a physicist possibly do with, say, the Banach-Tarski theorem (1924), which (via the Axiom of Choice) says that a solid ball of unit radius can be cut into five pieces, two of which form a solid ball of unit radius and the other three of which form a second such ball? Says author Gribbin: "Readers familiar with modern particle physics may be able to guess what is coming." In particular, "the rules governing the behaviour of these mathematical sets and subsets are formally exactly the same as the rules which describe the behaviour of quarks and gluons in quantum chromodynamics." A contrary view is that perhaps gluons and quarks aren't real after all but are just part of an model that reflects current scientific culture and the extensive training of particle physicists in mathematics!

NEWS AND LETTERS

TWENTYSECOND ANNUAL USA MATHEMATICAL OLYMPIAD PROBLEMS AND SOLUTIONS

1. For each integer $n \geq 2$, determine, with proof, which of the two positive real numbers a and b satisfying

$$a^n = a + 1, \qquad b^{2n} = b + 3a$$

is larger.

Solution. We shall prove that $a > b$ for every $n \geq 2$. Clearly $a \neq 1$, so $(a+1)^2 > 4a$. Suppose $b \geq a$. Then we have the contradictory requirements

$$\frac{b}{a} \leq \left(\frac{b}{a}\right)^{2n}$$

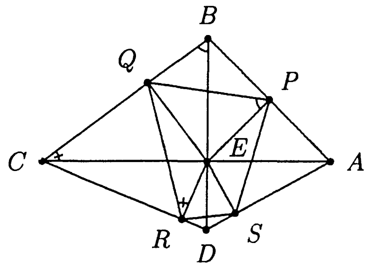
and

$$\left(\frac{b}{a}\right)^{2n} = \frac{b+3a}{(a+1)^2} < \frac{b+3a}{4a} \leq \frac{b}{a}.$$

Hence $a > b$. □

2. Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

Solution. Perform the homothety with ratio $\frac{1}{2}$ and center E. This dilation maps the reflections of E about AB, BC, CD, DE to the orthogonal projections of E onto the respective sides. Label these projections P, Q, R, S, as shown. It suffices to prove that $PQRS$ is a cyclic quadrilateral.



Quadrilaterals $ESAP$, $EPBQ$, $EQCR$, $ERDS$ are cyclic since each has a pair of opposite right angles. Since $EPBQ$ is cyclic, $\angle EPQ = \angle EBQ$, and since $EQCR$ is cyclic, $\angle ERQ = \angle ECQ$. Noting that $\angle EBQ$ and $\angle ECQ$ are the acute angles of a right triangle (ΔBCE), we have

$$\angle EPQ + \angle ERQ = \angle EBQ + \angle ECQ = 90°.$$

In exactly the same way, $\angle EPS + \angle ERS = 90°$, and it follows that

$$\angle SPQ + \angle QRS = 180°.$$

Hence $PQRS$ is a cyclic quadrilateral. □

Another solution uses inversion. As in the first solution, we use the fact that the quadrilaterals $ESAP$, $EPBQ$, $EQCR$, $ERDS$ are cyclic. Let $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$, $\mathcal{C}_D$ denote the corresponding circumcircles. Choose $k > 0$ arbitrarily and perform the inversion with respect to the circle of radius k centered at E. Consider the images under this inversion of $\mathcal{C}_A$ and $\overleftrightarrow{DB}$. Since the common point E is sent to infinity and $\overleftrightarrow{DB}$ is mapped to itself, it follows that the image of $\mathcal{C}_A$ is a line which is parallel to $\overleftrightarrow{DB}$. Similarly, the image of $\mathcal{C}_C$ is a line which is parallel to $\overleftrightarrow{DB}$, while the images of $\mathcal{C}_B$ and $\mathcal{C}_D$ are lines which are parallel to $\overleftrightarrow{CA}$ and thus perpendicular to $\overleftrightarrow{DB}$. Let P', Q', R', S' denote the images of P, Q, R, S, respectively, under the given inversion. Since P, Q, R, S are, in addition to E, the points of intersection of $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$, $\mathcal{C}_D$, it follows that P', Q', R', S' are the points of intersection of the four lines just described, and therefore they are vertices of a rectangle. The given inversion maps the circumcircle of this rectangle to a circle through P, Q, R, S, completing the proof. □

3. Consider functions $f : [0,1] \to \mathbb{R}$ which satisfy (i) $f(x) \geq 0$ for all x in $[0,1]$, (ii) $f(1) = 1$, (iii) $f(x) + f(y) \leq f(x+y)$ whenever x, y, and $x+y$ are all in $[0,1]$. Find, with proof, the smallest constant c such that

$$f(x) \leq cx$$

for every function f satisfying (i)-(iii) and every x in $[0,1]$.

Solution. The smallest value $c = 2$. First we prove that if f satisfies (i)-(iii) then $f(x) \leq 2x$ for all $x \in [0,1]$. (In fact, $f(x) < 2x$ for $x \in (0,1]$.) Setting $y = 1 - x$ in (iii) and using (i) and (ii), we have

$$f(x) \leq f(x) + f(1-x) \leq f(1) = 1,$$

for $x \in [0,1]$. Since $f(0) \geq 0$ and $f(0) + f(1) \leq f(1)$, it follows that $f(0) = 0$. Setting $y = x$ in (iii), we find

$$2f(x) \leq f(2x), \qquad x \in \left[0, 2^{-1}\right].$$

More generally, for each $n \geq 0$,

$$2^n f(x) \leq f(2^n x), \qquad x \in \left[0, 2^{-n}\right].$$

This is trivial for $n = 0$ and we have just seen that it is true for $n = 1$. For $n > 1$, induction yields

$$\begin{aligned} 2^n f(x) &= 2 \cdot 2^{n-1} f(x) \\ &\leq 2f(2^{n-1}x) \\ &\leq f(2^n x), \qquad x \in \left[0, 2^{-n}\right]. \end{aligned}$$

Let $x \in (0,1]$ and choose $n \geq 0$ so that $2^{-(n+1)} < x \leq 2^{-n}$. Then

$$\begin{aligned} 2^n f(x) &\leq f(2^n x) \\ &\leq f(1) \\ &= 1 \\ &< 2^{n+1} x, \end{aligned}$$

so $f(x) < 2x$.

To see that no smaller value of c will do, consider the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

This function clearly satisfies (i) and (ii). To see that it satisfies (iii), suppose x, y and $x + y$ are all in $[0,1]$, and note that we may assume by symmetry that $x < \frac{1}{2}$. Then since f is nondecreasing, $f(x) + f(y) = f(y) \leq f(x+y)$. If $0 \leq c < 2$,

$$f\left(1 - \frac{c}{4}\right) = 1$$

and

$$c\left(1 - \frac{c}{4}\right) = 1 - \left(1 - \frac{c}{2}\right)^2 < 1,$$

so $f(x) \leq cx$ fails for $x = 1 - c/4$. □

4. Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

Solution. Since $f_{n-1} + f_{n-2}$ is even and f_n is its largest odd divisor,

$$f_n \leq \frac{f_{n-1} + f_{n-2}}{2}, \qquad n \geq 3.$$

Hence $f_n \leq \max(f_{n-1}, f_{n-2})$, with equality if and only if $f_{n-1} = f_{n-2}$. For $k \geq 1$ let $c_k = \max(f_{2k}, f_{2k-1})$. Then

$$\begin{aligned} f_{2k+1} &\leq \max(f_{2k}, f_{2k-1}) = c_k \quad \text{and} \\ f_{2k+2} &\leq \max(f_{2k+1}, f_{2k}) \leq c_k, \end{aligned}$$

so $c_{k+1} \leq c_k$ with equality if and only if $f_{2k} = f_{2k-1}$. Since (c_n) is a nonincreasing sequence of positive integers, it is eventually constant and thus so is (f_n). From the recursive definition of (f_n), it follows that any common divisor of f_{n-1} and f_n also divides f_{n+1}. Hence for all sufficiently large n,

$$\begin{aligned} f_n &= \mathrm{GCD}(f_{n-1}, f_n) \\ &= \mathrm{GCD}(f_{n-2}, f_{n-1}) \\ \cdots &= \mathrm{GCD}(a, b). \ \square \end{aligned}$$

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \ldots$. (Such a sequence is said to be *log concave.*) Show that for each $n > 1$,

$$\frac{a_0 + \cdots + a_n}{n+1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \frac{a_0 + \cdots + a_{n-1}}{n} \cdot \frac{a_1 + \cdots + a_n}{n}.$$

Solution.

The fact that (a_k) is log concave yields

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{n-2}}{a_{n-1}} \leq \frac{a_{n-1}}{a_n},$$

and thus

$$a_0a_n \le a_1a_{n-1} \le a_2a_{n-2} \le \cdots . \qquad (1)$$

Set $s = a_1 + a_2 + \cdots + a_{n-1}$ and write the inequality to be proved as

$$n^2(s + a_0 + a_n)s \ge (n^2 - 1)(s + a_0)(s + a_n).$$

Canceling common terms, we obtain the equivalent inequality

$$(s + a_0)(s + a_n) \ge n^2\, a_0a_n. \qquad (2)$$

To prove (2), we first use the arithmetic mean - geometric mean inequality together with (1) to conclude

$$\begin{aligned} s &= \sum_{k=1}^{n-1}\left(\frac{a_k + a_{n-k}}{2}\right) \\ &\ge \sum_{k=1}^{n-1}\sqrt{a_ka_{n-k}} \\ &\ge (n-1)\sqrt{a_0a_n}. \end{aligned}$$

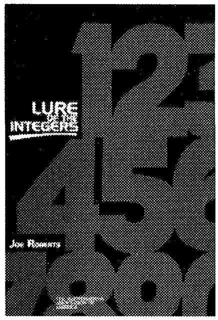
Then since $a_0 + a_n \ge 2\sqrt{a_0a_n}$, we have

$$\begin{aligned} (s + a_0)(s + a_n) &\ge (s + \sqrt{a_0a_n})^2 \\ &\ge n^2\, a_0a_n. \end{aligned}$$

Solutions were prepared by Cecil Rousseau, Memphis State University, Memphis, TN 38152.

LURE OF THE INTEGERS

Joe Roberts



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